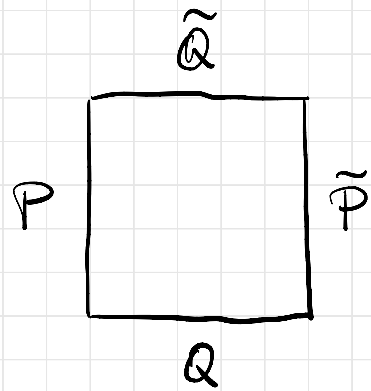
A bijective approach to KPZ solvable models

(Lecture 3)

Last time we defined the skew RSK map

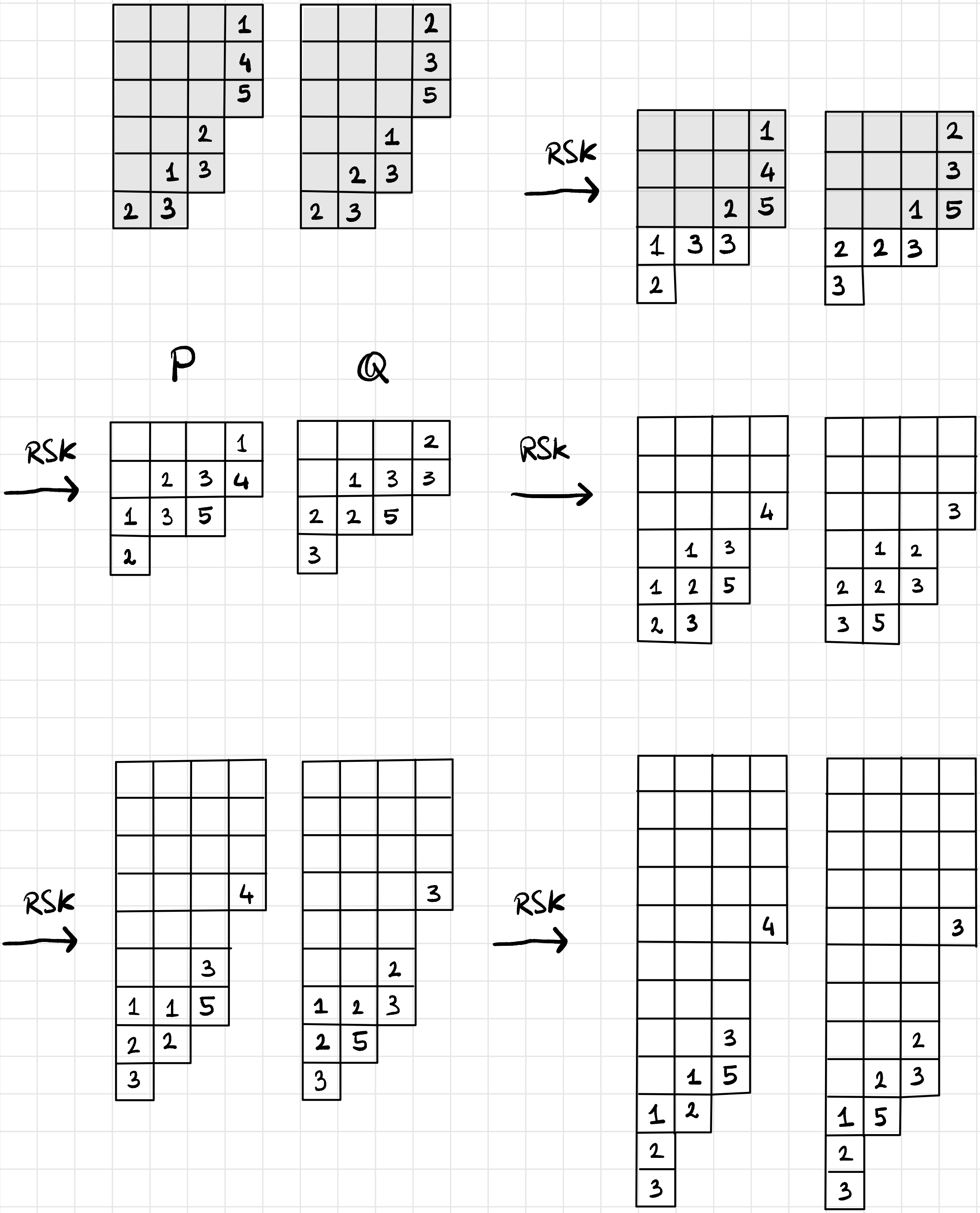
$$\mathrm{RSK}(P, Q) = (\tilde{P}, \tilde{Q})$$



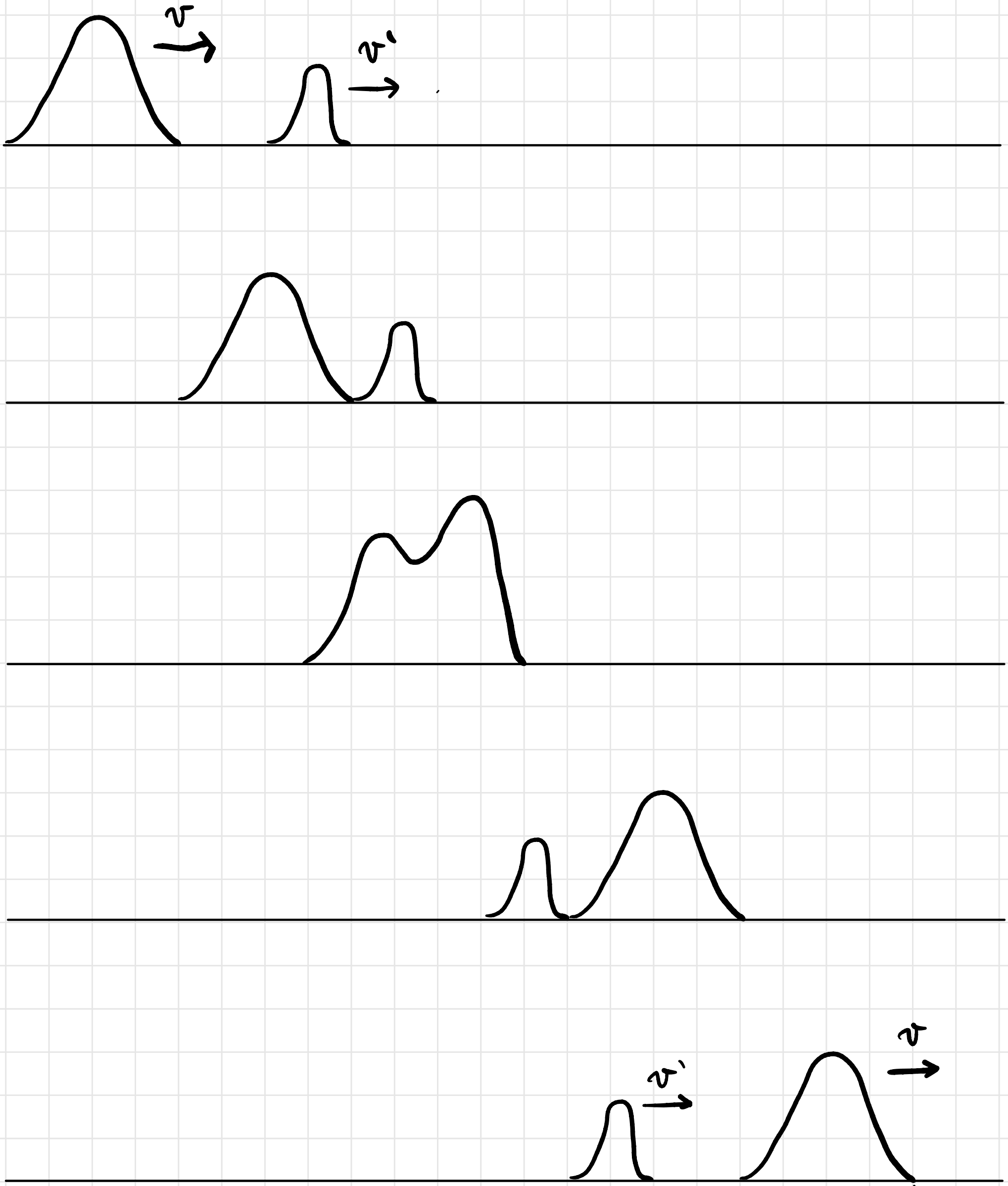
and the skew RSK dynamics

$$(P^{(t)}, Q^{(t)}) = \mathrm{RSK}^t(P, Q)$$

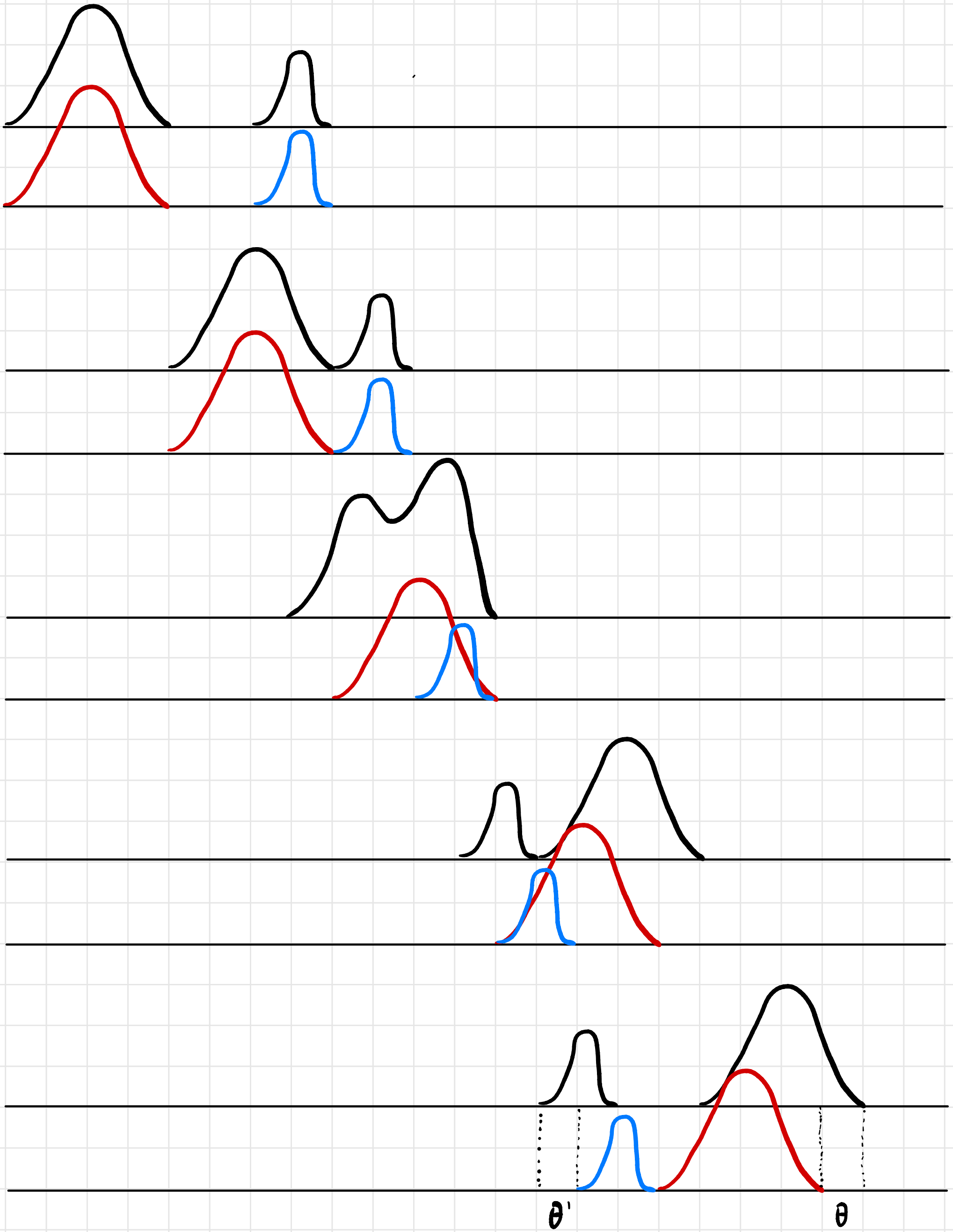
with initial conditions (P, Q).


RSK
P
Q
RSK
RSK
RSK
RSK

Schematical description of solitonic systems



Schematical description of solitonic systems



θ, θ' = phase shifts

The existence of (infinitely many) conservation laws makes the dynamics exactly solvable.

In other words there exists a (non-linear) map Υ

physical coordinate   quasi-momenta   locations

$$
\begin{tikzcd}
y(t) \arrow[r, "\Upsilon"] \arrow[d] & \left(\{p_i\}_i \, ; \, \{x_i\}_i \right) \arrow[d] \\
y(t+T) & \left(\{p_i\} \, ; \, \{x_i + T p_i\} \right) \arrow[l, "\Upsilon^{-1}"']
\end{tikzcd}
$$

In our case the physical coordinates are

$$y(t) = (P^{(t)}, Q^{(t)})$$

while we have seen that solitons are

$$\{P_i\} \longrightarrow (V, W) = \text{vertically strict tableaux of shape } \mu$$

$$\{x_i\} \longrightarrow (K_1, \dots, K_{\mu_1}) \quad K_i \in \mathbb{N}$$

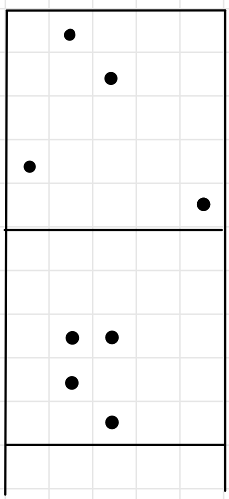
with possibly some conditions

$$
\begin{array}{ccccc}
P & Q & & V & W & K \\
\begin{array}{|c|c|c|c|}\hline & & & 1 \\ \hline & 2 & 3 & 4 \\ \hline 1 & 3 & 5 \\ \cline{1-3} 2 \\ \cline{1-1}\end{array}
&
\begin{array}{|c|c|c|c|}\hline & & & 2 \\ \hline & 1 & 3 & 3 \\ \hline 2 & 2 & 5 \\ \cline{1-3} 3 \\ \cline{1-1}\end{array}
&
\overset{\Upsilon}{\rightsquigarrow}
&
\begin{array}{|c|c|c|c|}\hline 1 & 1 & 3 & 4 \\ \hline 2 & 2 & 5 \\ \cline{1-3} 3 \\ \cline{1-1}\end{array}
&
\begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 3 \\ \hline 2 & 5 & 3 \\ \cline{1-3} 3 \\ \cline{1-1}\end{array}
&
??
\end{array}
$$

Question : Can we characterize the slope μ from tableaux (P, Q) ?

$$
\begin{array}{ccccc}
P & Q & & V & W & \kappa \\
\begin{array}{|c|c|c|c|}\hline & & & 1 \\ \hline & 2 & 3 & 4 \\ \hline 1 & 3 & 5 \\ \cline{1-3} 2 \\ \cline{1-1}\end{array}
&
\begin{array}{|c|c|c|c|}\hline & & & 2 \\ \hline & 1 & 3 & 3 \\ \hline 2 & 2 & 5 \\ \cline{1-3} 3 \\ \cline{1-1}\end{array}
&
\overset{\Upsilon}{\rightsquigarrow}
&
\begin{array}{|c|c|c|c|}\hline 1 & 1 & 3 & 4 \\ \hline 2 & 2 & 5 \\ \cline{1-3} 3 \\ \cline{1-1}\end{array}
&
\begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 3 \\ \hline 2 & 5 & 3 \\ \cline{1-3} 3 \\ \cline{1-1}\end{array}
&
??
\end{array}
$$

$$\updownarrow$$



$\nu = \emptyset$

$$\ell_1 = 4 = \mu_1$$

$$\ell_2 = 7 = \mu_1 + \mu_2$$

$$\ell_3 = 8 = \mu_1 + \mu_2 + \mu_3$$

Question : Can we characterize the symmetries of the skew RSK dynamics ?

Kashiwara operators $\tilde{e}_i, \tilde{f}_i$

We will consider words / tableaux (resp. matrices) with entries in $\{1, \dots, n\}$ (resp n rows / col.).

Action on words:

$$w = 4\,2\,3\,2\,1\,2\,3\,1\,4\,3\,3\,2\,1\,2\,4\,1\,2\,3\,3$$

$$)\,(\,)\quad)\,(\quad (\,(\,)\quad)\quad)\,(\,($$

$$)\ \cancel{()}\quad)\ \cancel{(}\quad \cancel{(}\ \cancel{()}\quad \cancel{)}\quad \cancel{)}\ (($$

$$\tilde{e}_2(w) = 4\,2\,3\,2\,1\,2\,3\,1\,4\,3\,3\,2\,1\,2\,4\,1\,2\,2\,3$$

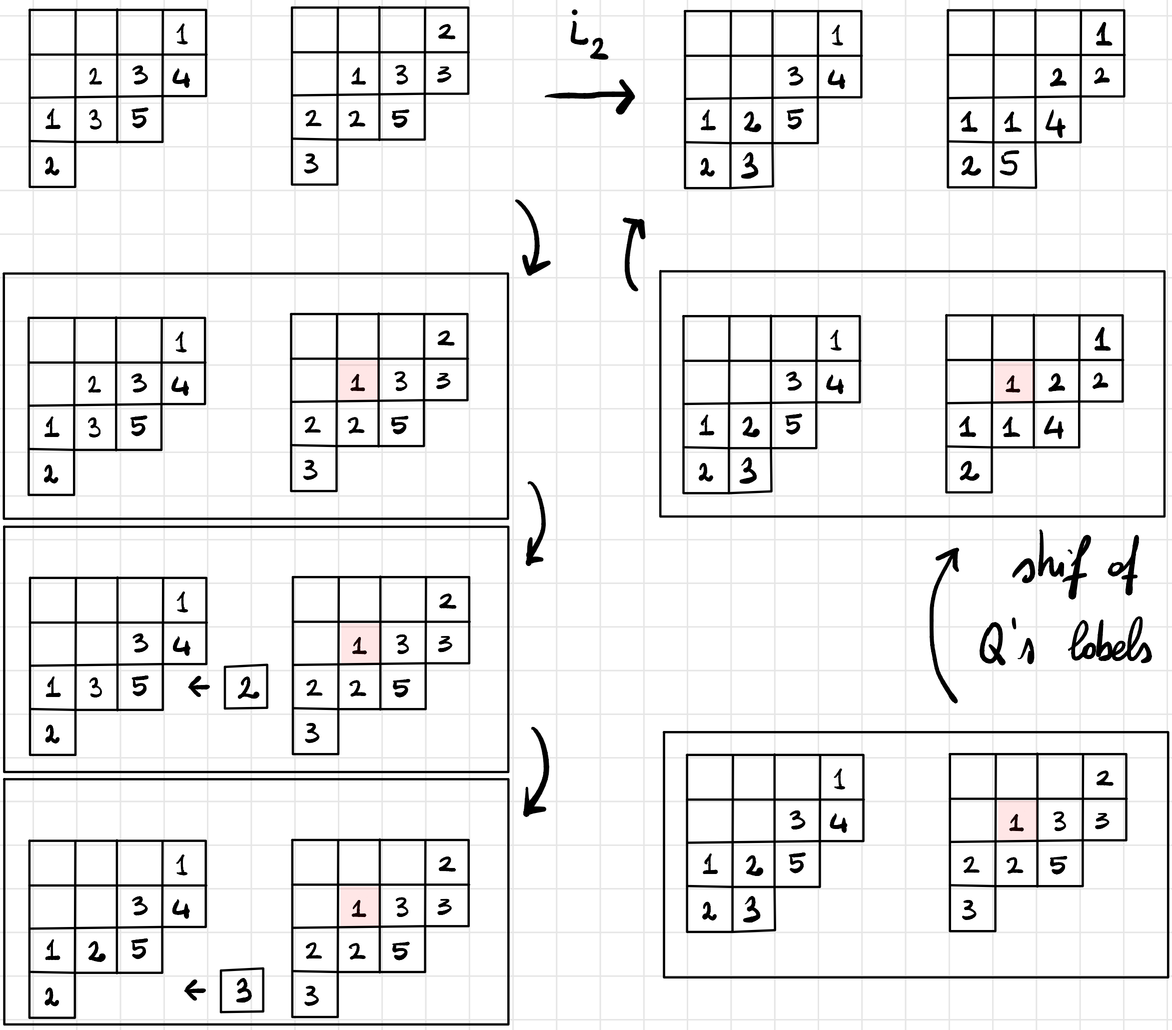
$$\tilde{f}_2(w) = 4\,2\,3\,2\,1\,3\,3\,1\,4\,3\,3\,2\,1\,2\,4\,1\,2\,3\,3$$

Action on tableaux:

$$\begin{array}{|c|c|c|c|c|} \hline & & & 1 & 2 \\ \hline & 1 & 2 & 2 \\ \cline{1-4} 2 & 3 & 3 \\ \cline{1-3} \end{array} \xrightarrow{\tilde{e}_1} \begin{array}{|c|c|c|c|c|} \hline & & & 1 & 2 \\ \hline & 1 & 1 & 2 \\ \cline{1-4} 2 & 3 & 3 \\ \cline{1-3} \end{array}$$

$$\cancel{()}\ (\ \cancel{()}\ ($$

i_ε = internal insertion with cycling , $\varepsilon = 1, 2$



- When there are multiple 1's in Q proceed from left to the right
- $i_1(P, Q) = \mathrm{swap}(i_2(Q, P))$

We can equip the set of pairs (P, Q) with an "affine bicrystal" structure

$$\tilde{E}_i^{(1)} = \tilde{e}_i \times 1 \qquad \tilde{F}_i^{(1)} = \tilde{f}_i \times 1$$

$$\tilde{E}_i^{(2)} = 1 \times \tilde{e}_i \qquad \tilde{F}_i^{(2)} = 1 \times \tilde{f}_i$$

for $i = 1, \dots, n-1$

$$\tilde{E}_0^{(\varepsilon)} = \iota_\varepsilon \circ \tilde{E}_1^{(\varepsilon)} \circ \iota_\varepsilon^{-1}$$

$$\tilde{F}_0^{(\varepsilon)} = \iota_\varepsilon \circ \tilde{F}_1^{(\varepsilon)} \circ \iota_\varepsilon^{-1}$$

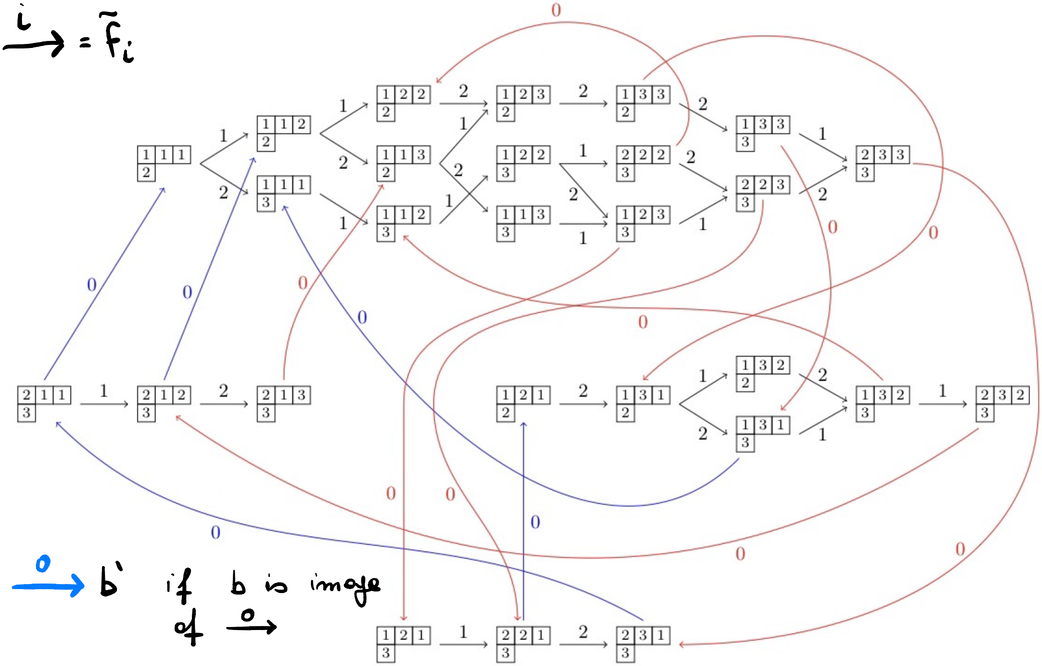
for $i = 0$

Theorem The skew RSK map commutes with the affine bicrystal structure on (P,Q). Let h be any composition of $E_i^{(\varepsilon)}$, $F_i^{(\varepsilon)}$ $\varepsilon = 1, 2$, $i = 0, \dots, n-1$.
Then, for all (P,Q), we have

$$h\big(\mathrm{RSK}(P,Q)\big) = \mathrm{RSK}\big(h(P,Q)\big).$$

Theorem The affine bicrystal structure on (P,Q) induces an affine bicrystal structure on (V,W).
Moreover the latter is isomorphic to a product of two affine crystal structures on V and W.

$\xrightarrow{i} = \tilde{f}_i$



$b \xrightarrow{0} b'$ if b is image of $\xrightarrow{0}$

Facts:

- The induced graph is a Kirillov-Reshetikhin crystal
- Demazure subgraph : erase $\xrightarrow{0}$
- The Demazure subgraph is connected.

[Fourier-Schilling-Shimozono]

Def: $\mathcal{L}_v : V \longrightarrow Yom(\mu)$ Leading map

Def: $\mathcal{H}(v) = \#\tilde{f}_0 - \#\tilde{e}_0$ in $\mathcal{L}_v$ Intrinsic Energy

Theorem [Sanderson]

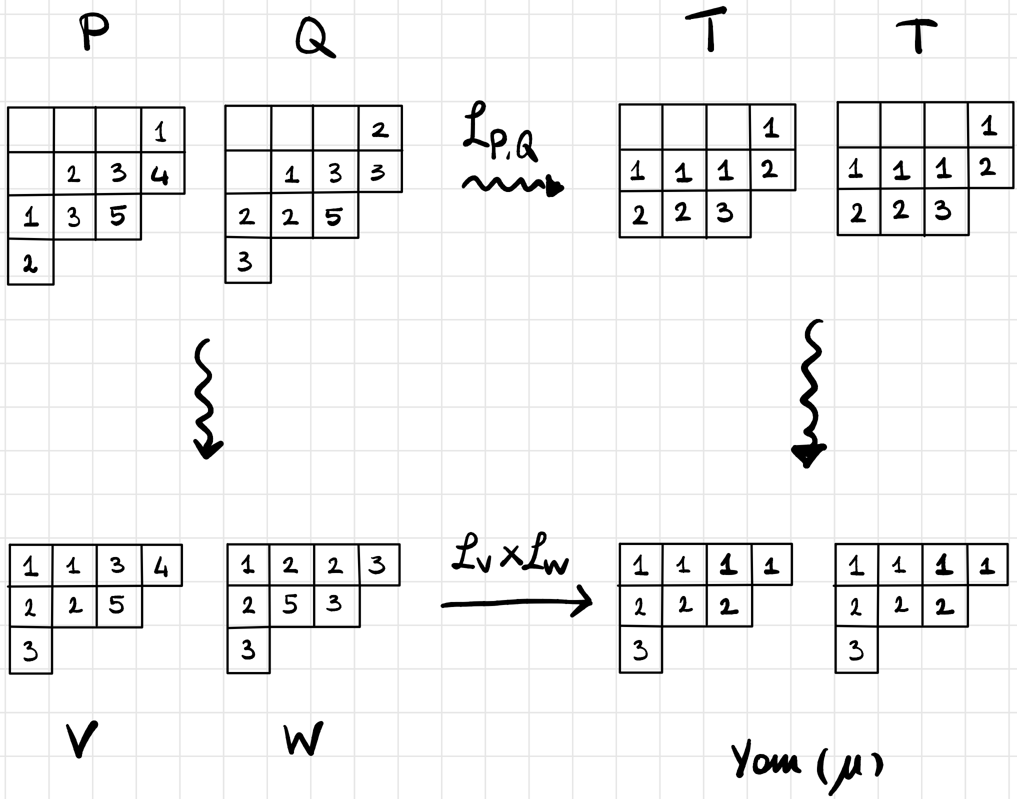
$$\sum_{\substack{V \text{ vertically} \\ \text{strict tableau} \\ \text{of shape } \mu}} q^{\chi(V)} a^{V} = P_\mu(a)$$

where P_μ is the q-Whittaker polynomial.

We can induce a Demazure crystal structure on pairs (P, Q).

Def $F_0^{(\varepsilon)} : (P, Q) \longrightarrow (\tilde{P}, \tilde{Q})$ is a Demazure arrow if (P, Q) is image of $F_0^{(\varepsilon)}$.

Def. $\mathcal{L}_{P,Q}(P,Q) = (T,T)$ is a leading map if it is the pre-image of a product of leading maps for V, W



Lemma Assume $P, Q \sim \lambda/\xi$ and $T \sim \tilde{\lambda}/\tilde{\xi}$

Then

$$|\xi| - |\tilde{\xi}| = \mathcal{H}(V) + \mathcal{H}(W).$$

We can characterize the image of leading maps

Def T is a leading tableaux if whenever it has k i-cells at row r, then it has at least k $(i-1)$-cells at row $r-1$, for each $k, r > 0$ and $i = 2, \dots n$.

Theorem There exists a bijection

$\{\text{Leading tableaux } T\}$ $\qquad$ $\{(\mu; \kappa; \nu)\}$

μ, ν partitions

$\kappa = (\kappa_1, \dots, \kappa_{\mu_1})$

$: \kappa_i \geq \kappa_{i+1}$ if $\mu_i = \mu_{i+1}$

$$\begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 \\ \cline{1-3} \end{array} \quad \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 \\ \cline{1-3} \end{array} \longleftrightarrow \left(\ydiagram{4,3,1} \; ; (0,1,1,1) ; \emptyset \right)$$

Theorem We have constructed a bijection

$$(P, Q) \xleftrightarrow{\gamma} (V, W; \kappa, \nu)$$

such that

1) $|\rho| = \mathcal{H}(V) + \mathcal{H}(W) + |\kappa| + |\nu|$

2) $\lambda_1 = \mu_1 + \nu_1$

3) If $(\{w_{ij}^{(k)}\}; \nu) \leftrightarrow (P, Q) \Rightarrow \ell_k(w) = \mu_1 + \dots + \mu_k$

$$\left(\begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & 2 & 3 & 4 \\ \hline 1 & 3 & 5 \\ \cline{1-3} 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|c|} \hline & & & 2 \\ \hline & 1 & 3 & 3 \\ \hline 2 & 2 & 5 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array} \right)$$

$$\updownarrow$$

$$\left(\begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & 4 \\ \hline 2 & 2 & 5 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 & 3 \\ \hline 2 & 5 & 3 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array}, (0,1,1,1); \emptyset \right)$$

Cauchy Identity

$$\frac{1}{(q;q)_\infty} \prod_{i,j=1}^{n} \frac{1}{(a_i b_j ; q)_\infty}$$

$$= \sum_{\lambda, \rho} q^{|\rho|} s_{\lambda/\rho}(a)\, s_{\lambda/\rho}(b)$$

$$= \sum_{\lambda,\rho} \sum_{(P,Q)\sim\lambda/\rho} q^{|\rho|} a^{P} b^{Q}$$

$$= \sum_{\mu,\varkappa,\nu} \sum_{V,W\sim\mu} q^{\varkappa(V)+\varkappa(W)+|\varkappa|+|\nu|} a^{V} b^{W}$$

$$= \left(\sum_{\nu} q^{|\nu|}\right) \sum_{\mu} \left(\sum_{\varkappa} q^{|\varkappa|}\right)\left(\sum_{V} q^{\varkappa(V)} a^{V}\right)\left(\sum_{W} q^{\varkappa(W)} b^{W}\right)$$

$$= \frac{1}{(q;q)_\infty} \sum_{\mu} b_\mu(q)\, \mathcal{P}_\mu(a)\, \mathcal{P}_\mu(b).$$

Kawanaka - Littlewood identity

Forcing $P = Q$ in Υ we obtain a new bijection

$$P \longleftrightarrow (V; \kappa; \nu)$$

$$\frac{1}{(z^2q;q^2)_\infty}\frac{1}{(q^2;q^2)_\infty}\prod_{i=1}^{n}\frac{1}{(a_iz;q)_\infty}\prod_{i<j}\frac{1}{(x_ix_j;q^2)_\infty}$$

$$= \sum_{\lambda,\rho} z^{\mathrm{odd}(\lambda')+\mathrm{odd}(\rho')} q^{|\rho|} s_{\lambda/\rho}(a)$$

$$= \sum_{\lambda,\rho}\sum_{P} z^{\mathrm{odd}(\lambda')+\mathrm{odd}(\rho')} q^{|\rho|} a^{P}$$

$$= \sum_{\mu,\kappa,\nu} z^{\mathrm{odd}(\kappa)+\mathrm{odd}(\mu'+\kappa)+2\,\mathrm{odd}(\nu')} \times q^{|\kappa|+2\mathcal{H}(V)+|\nu|} a^{V}$$

$$= \left(\sum_{\nu} q^{|\nu|} z^{2\,\mathrm{odd}(\nu')}\right)\sum_{\mu}\left(\sum_{\kappa} z^{\mathrm{odd}(\kappa)+\mathrm{odd}(\mu'+\kappa)} q^{|\kappa|}\right) q^{2\mathcal{H}(V)} a^{V}$$

$$= \frac{1}{(z^2q;q^2)_\infty}\frac{1}{(q^2;q^2)_\infty}\sum_{\mu} b_\mu(z;q)\, P_\mu(a;q^2)$$