Bijective proof of Cauchy Identity for q-Whittaker polynomials

SLC — April 6th 2022

Matteo Mucciconi — based on a joint work with Takashi Imamura and Tomohiro Sasamoto

Cauchy Identities

$$\sum_{\lambda} b_{\lambda}(q,t) P_{\lambda}(x;q,t) P_{\lambda}(y;q,t) = \prod_{k\geq 0} \prod_{i,j} \frac{1 - t x_i y_j q^k}{1 - x_i y_j q^k}$$

- $P_{\lambda}(x;q,t) =$ Macdonald Polynomials, $b_{\lambda}(q,t)$ = explicit factor

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- The Cauchy Identity can be used to define $P_\lambda(x;q,t)$ [Macdonald's book]
- There are combinatorial definitions of Macdonald polynomials: tableaux expansion [Haglund-Haiman-Loher], Vertex models [Cantini-De Gier-Wheeler, Borodin-Wheeler, Garbali-Wheeler,...], alcove walks [Ram-Yip],...

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  - $t = 0$ YES : [Imamura-M-Sasamoto'21]
  - $q, t \neq 0$???

Cauchy Identity for Schur polynomials

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$$\sum_{\lambda} \sum_{P,Q \in \mathrm{SSYT}(\lambda)} x^P y^Q = \sum_{M \in \mathbb{M}_{n\times n}} \prod_{i,j=1}^{n} (x_i y_j)^{M_{i,j}}$$

Bijective proof:

$$(P, Q) \overset{\textbf{RSK}}{\longleftrightarrow} M$$

RSK correspondence

$$\left(M_{i,j}\right)_{i,j=1}^{n} \longleftrightarrow (P, Q)$$

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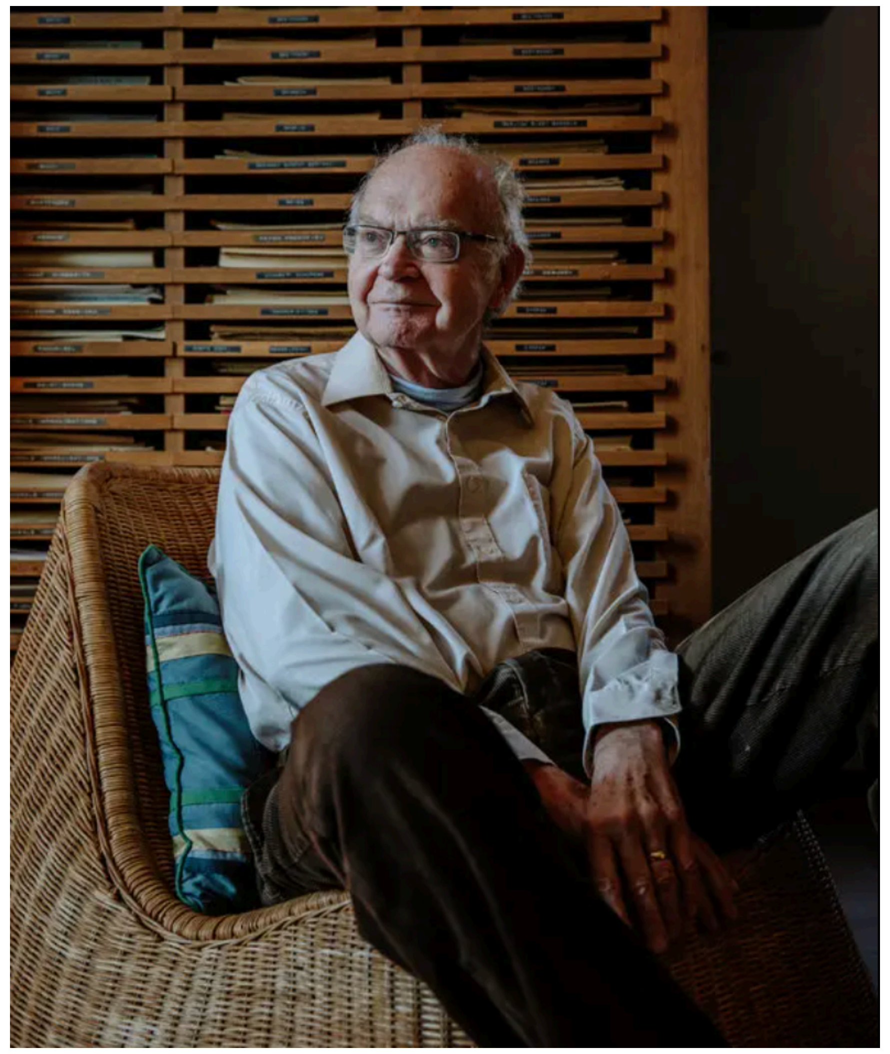
Robinson

Schensted



Knuth

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Example

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix} \longleftrightarrow ?, ?$$

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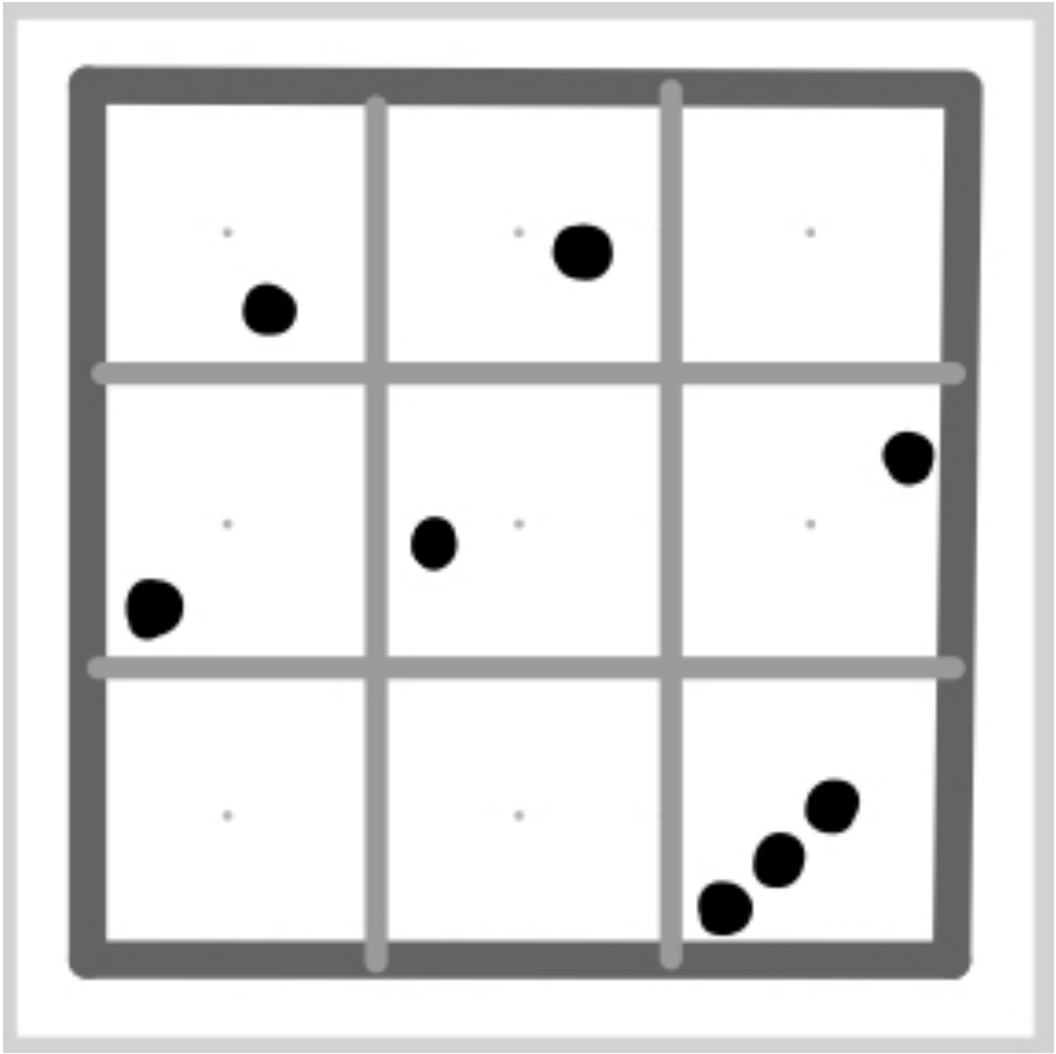

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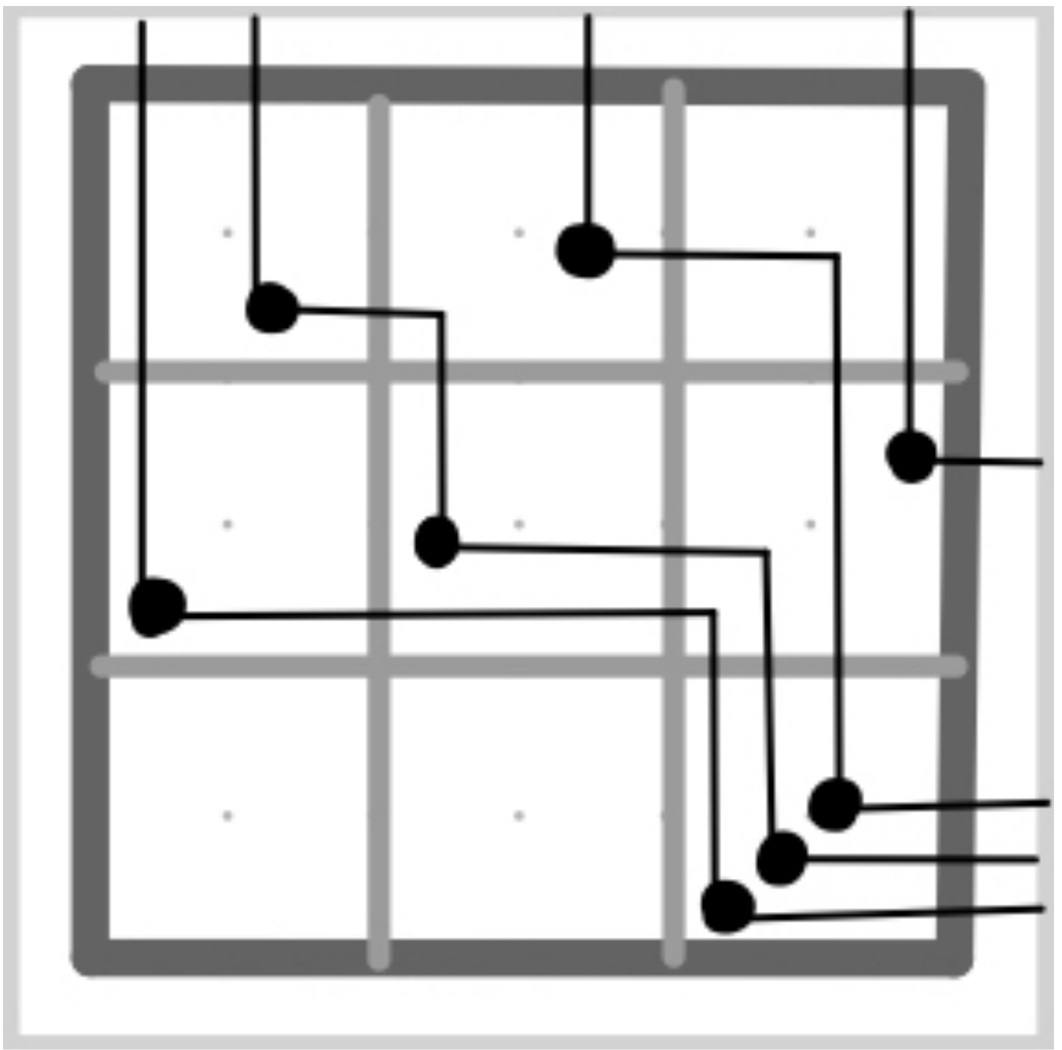

X.G. Viennot

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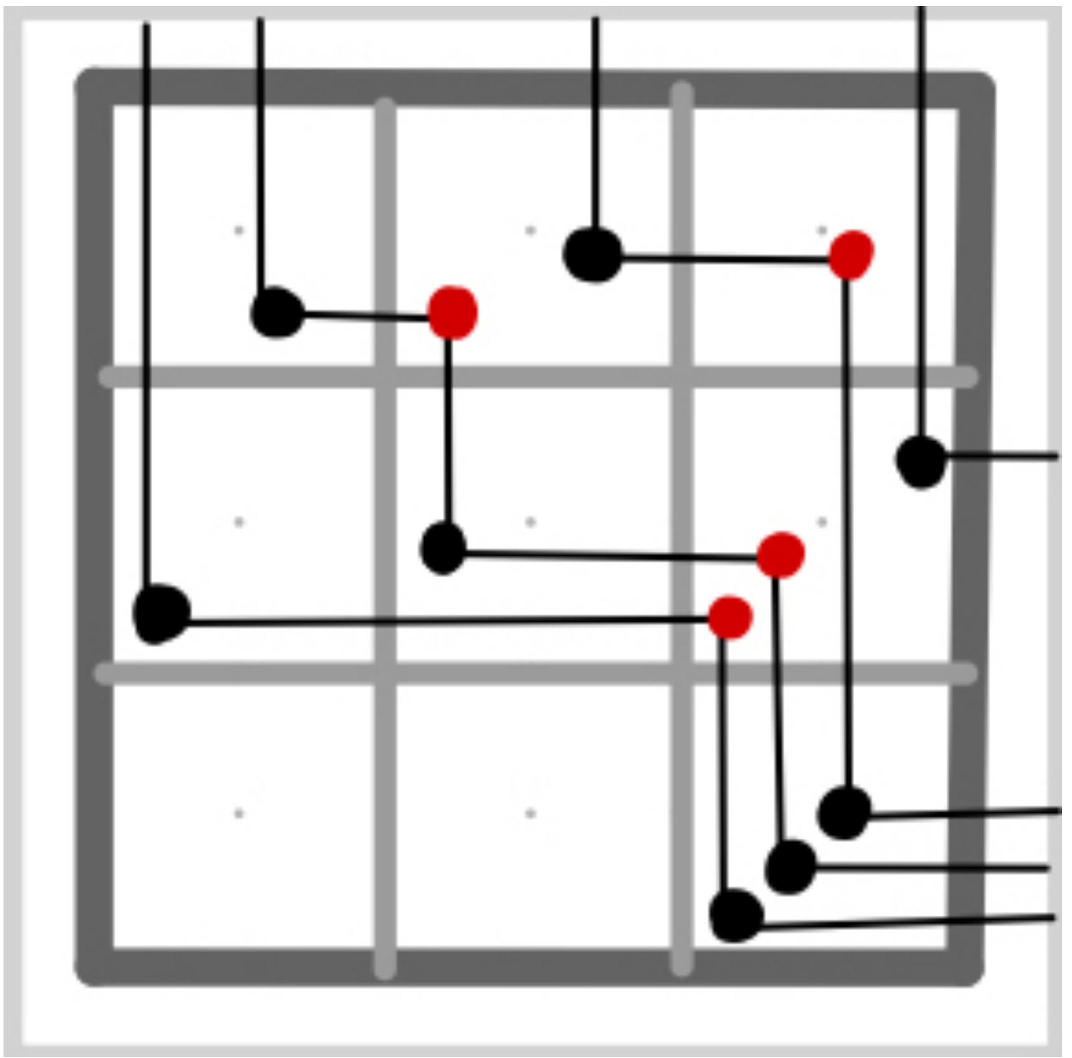

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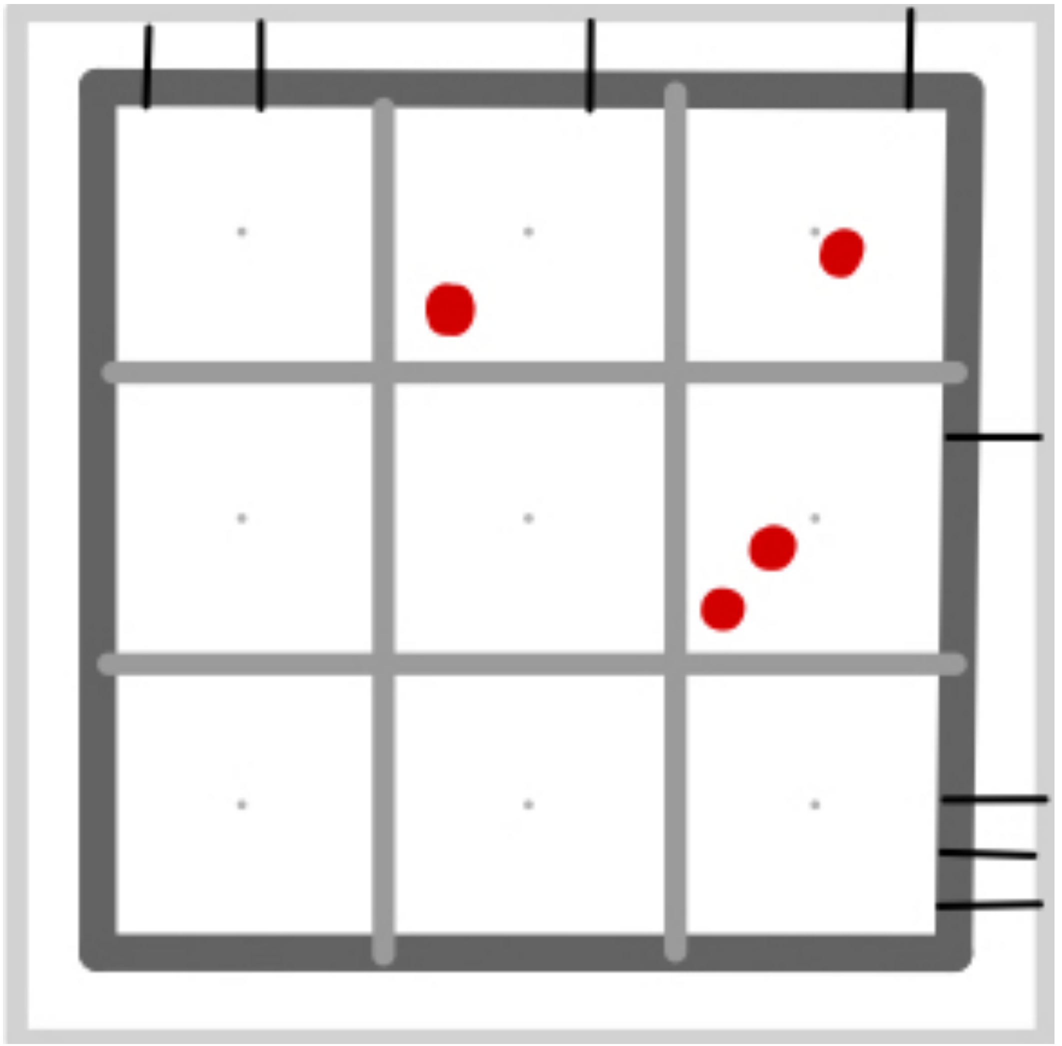

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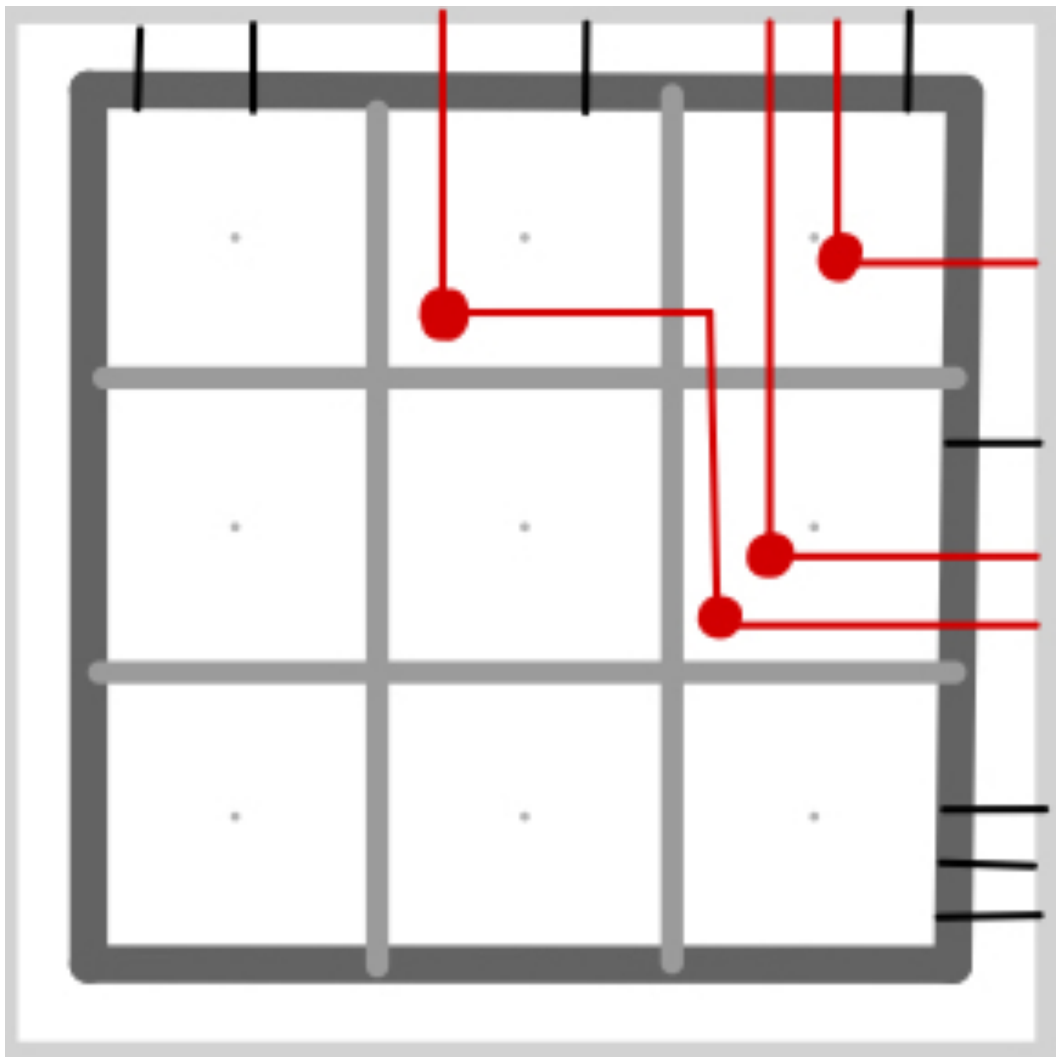

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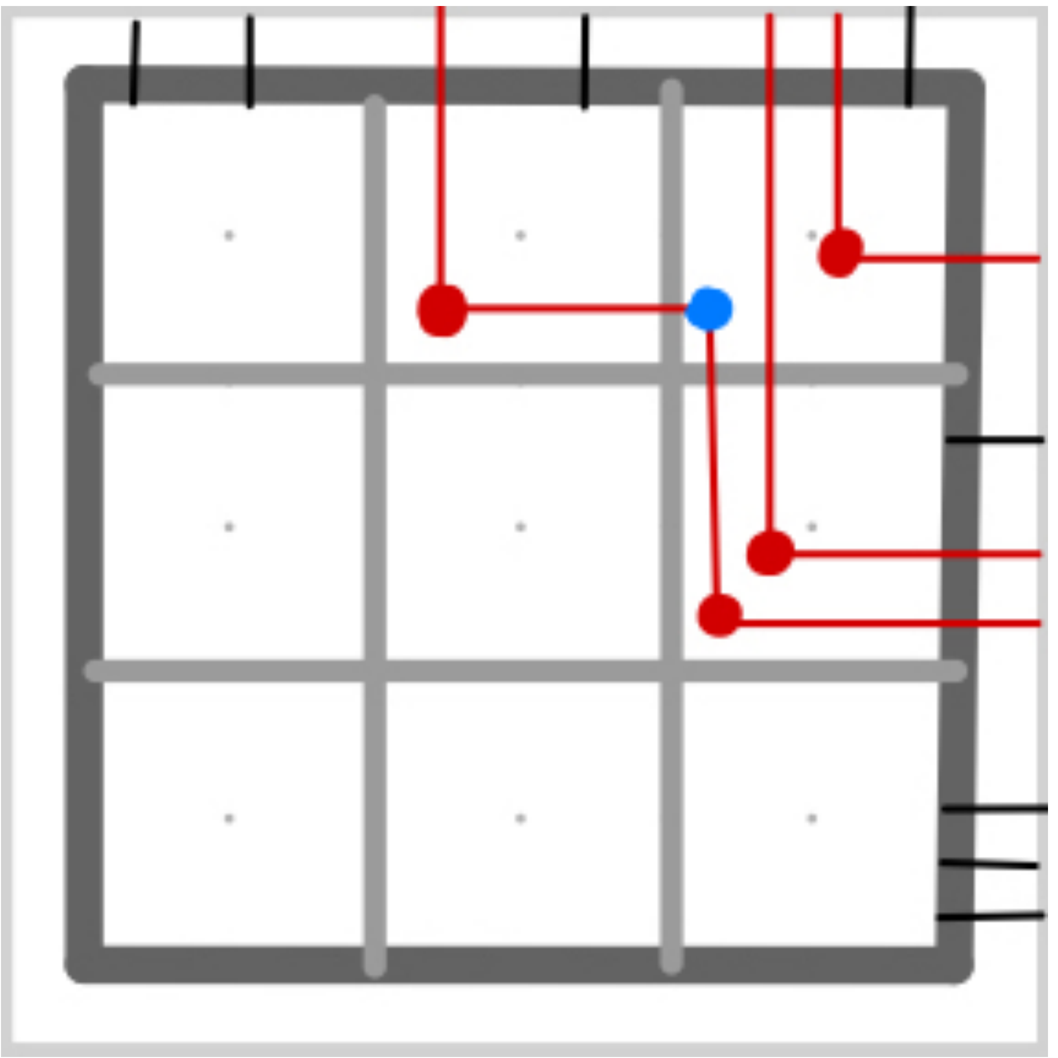

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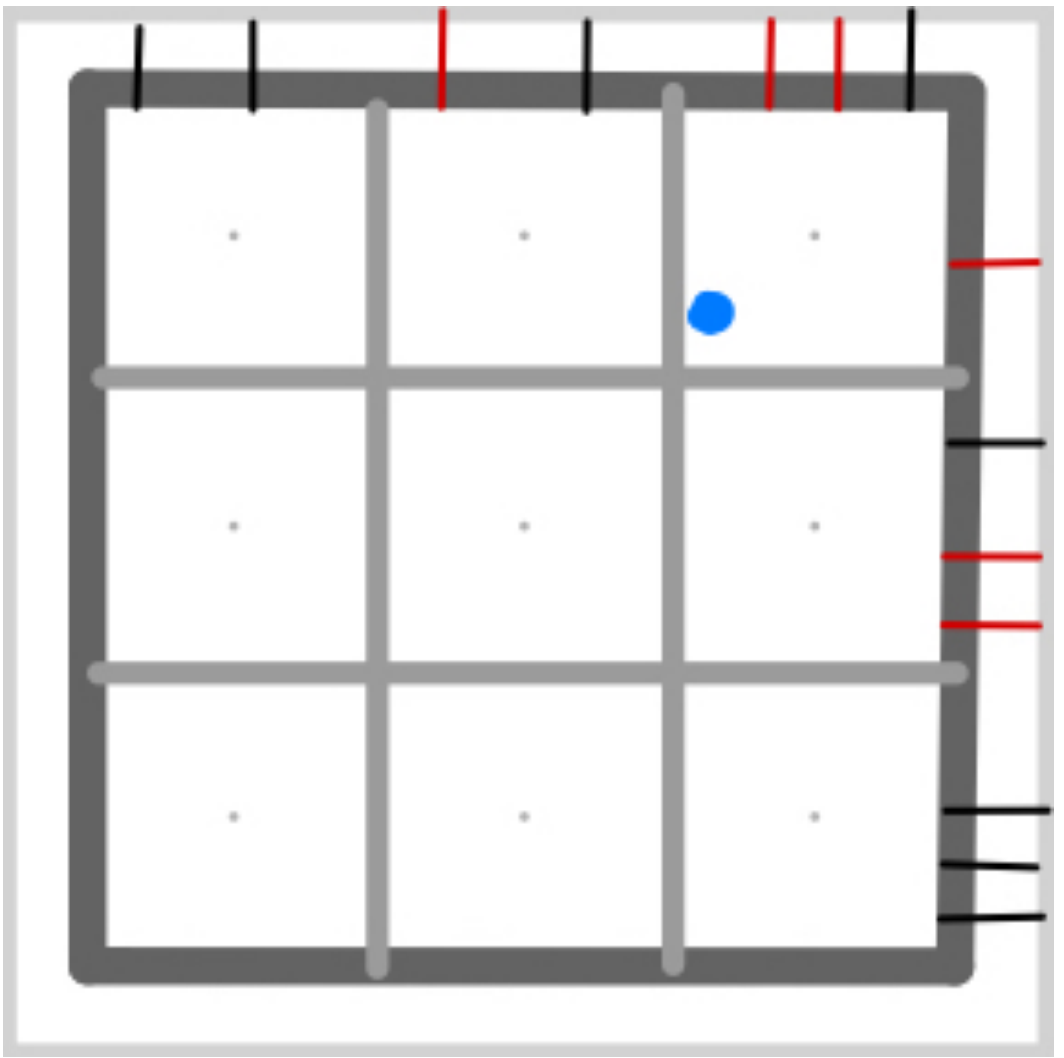

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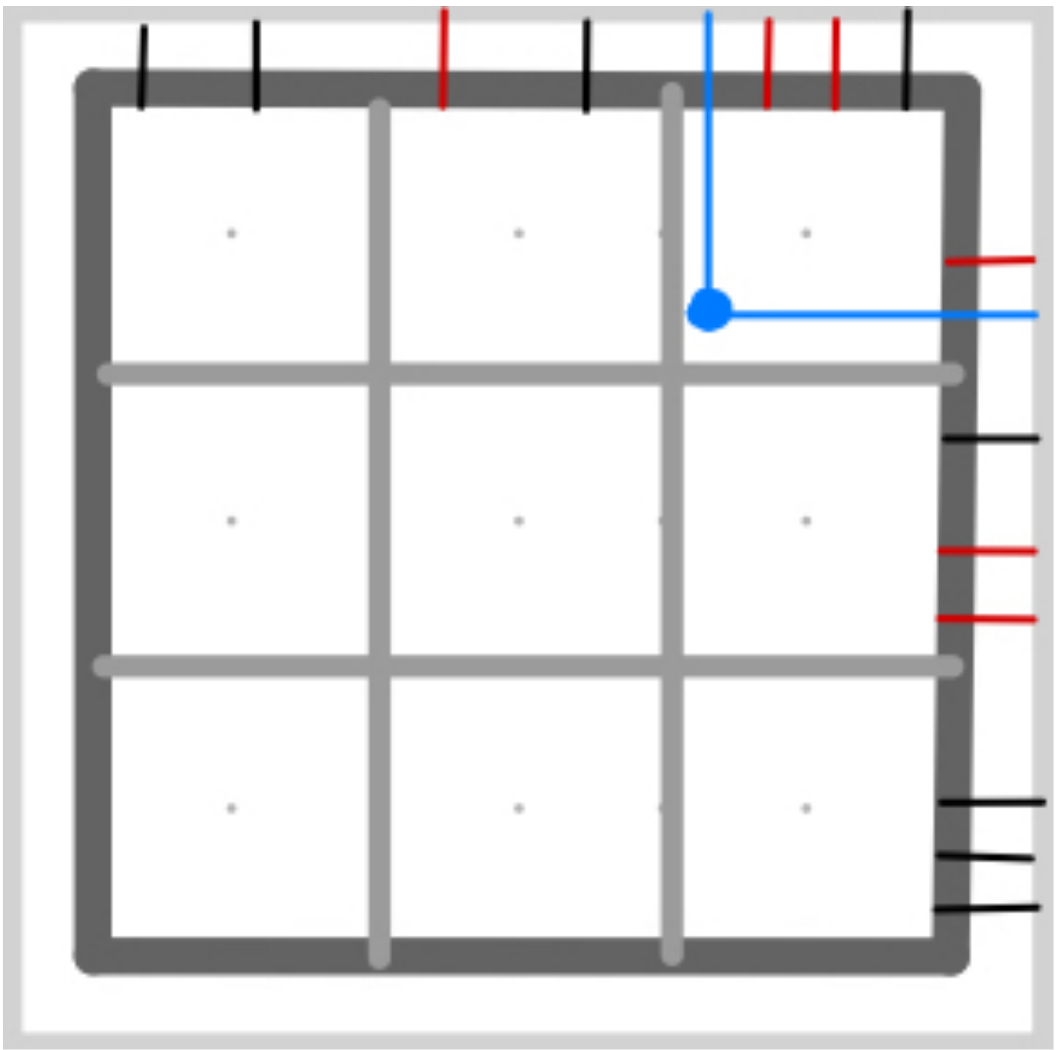

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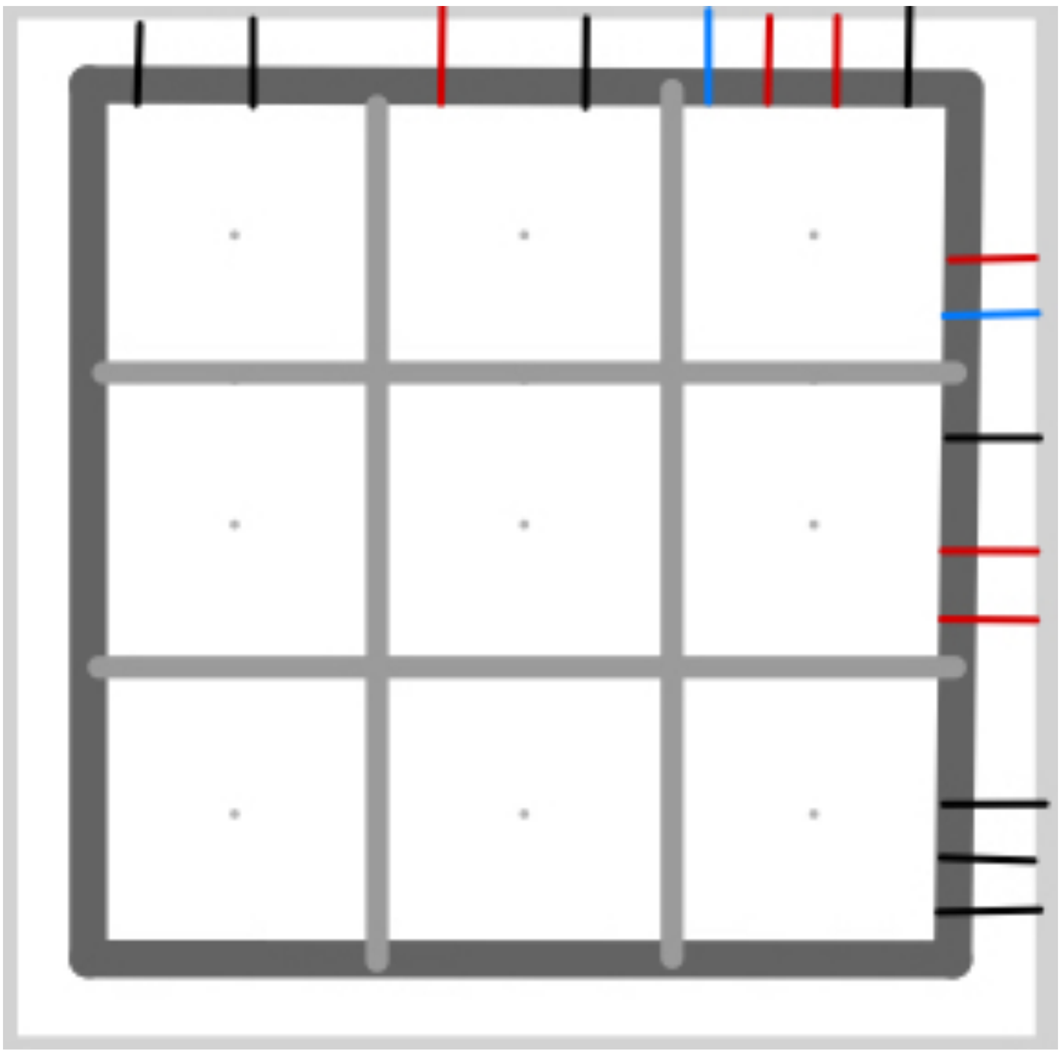

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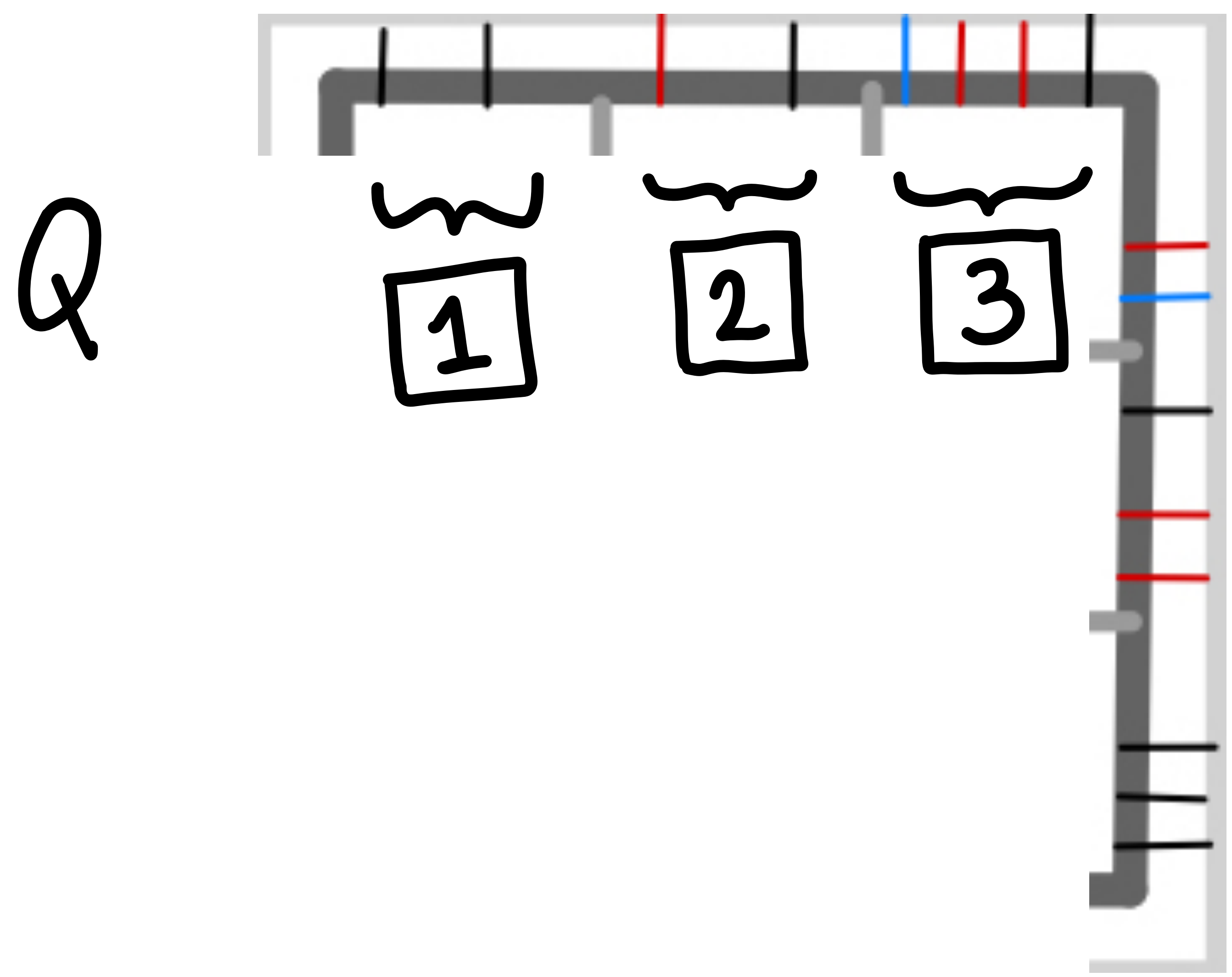

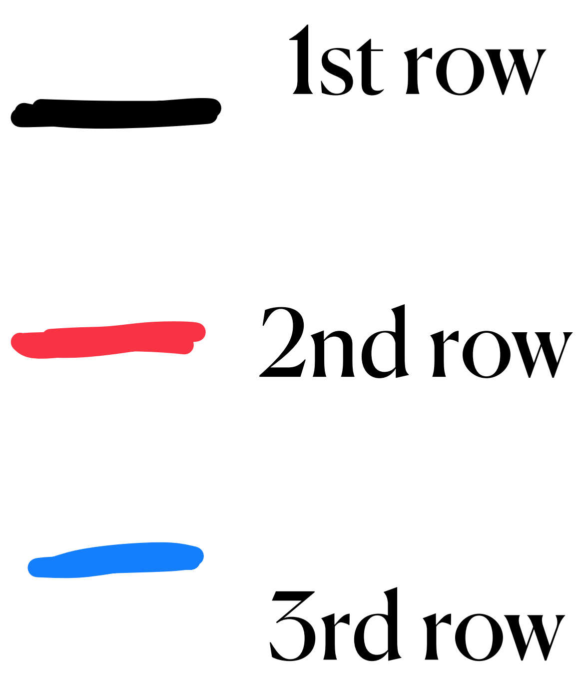

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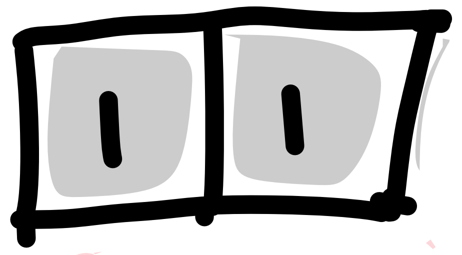

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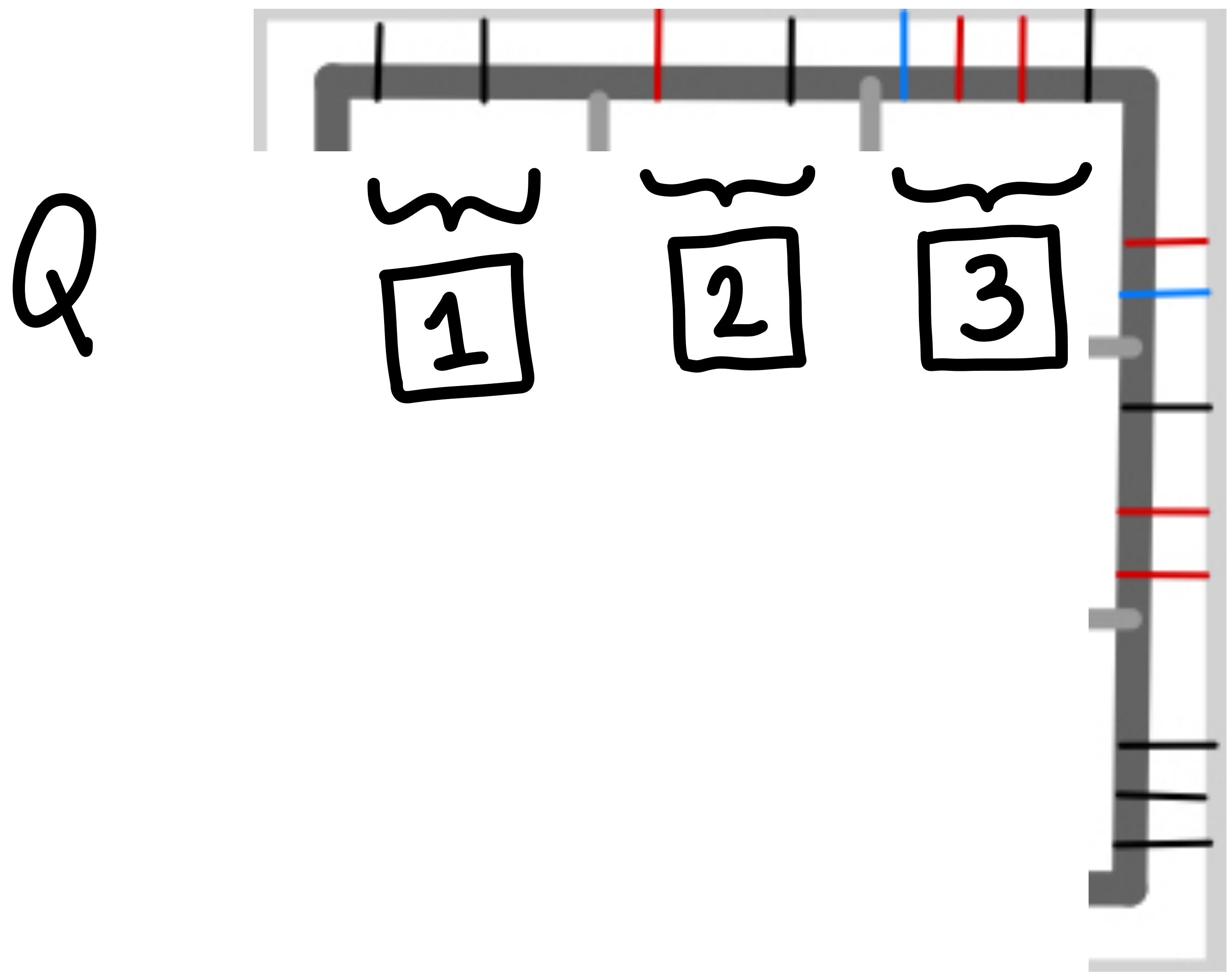

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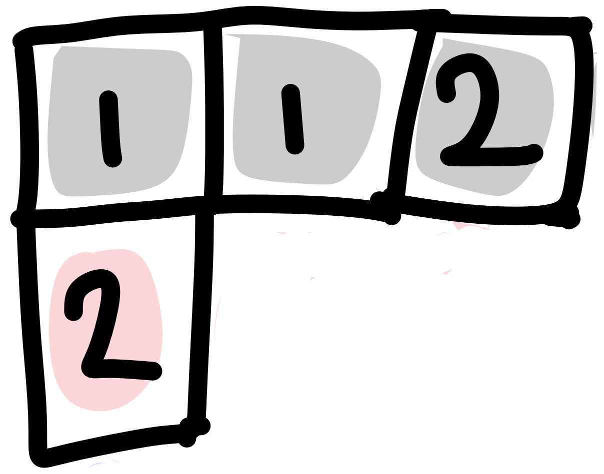

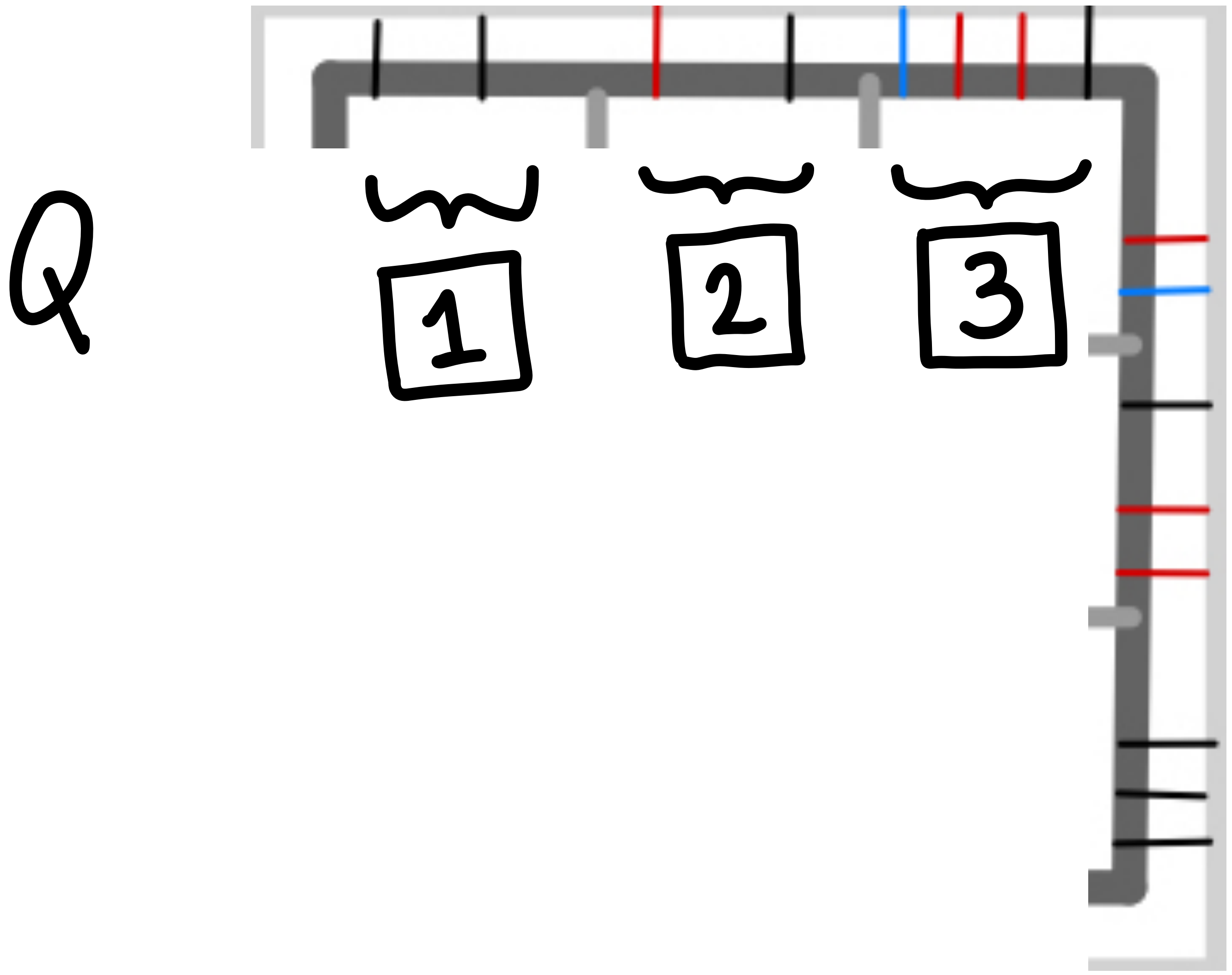

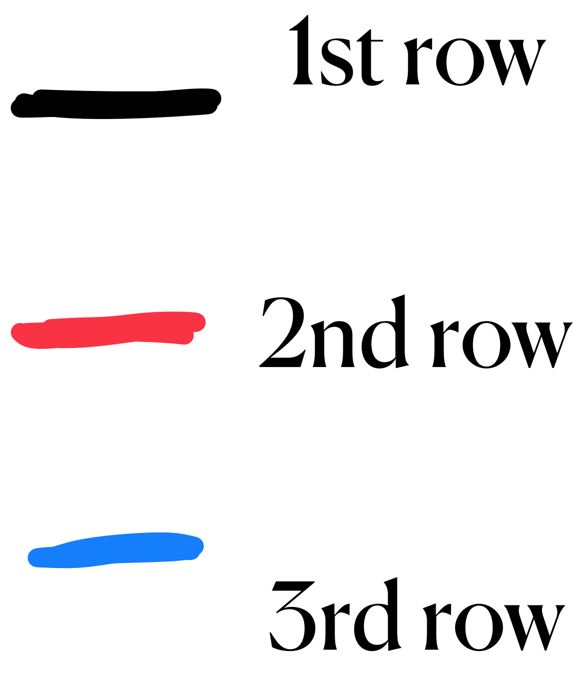

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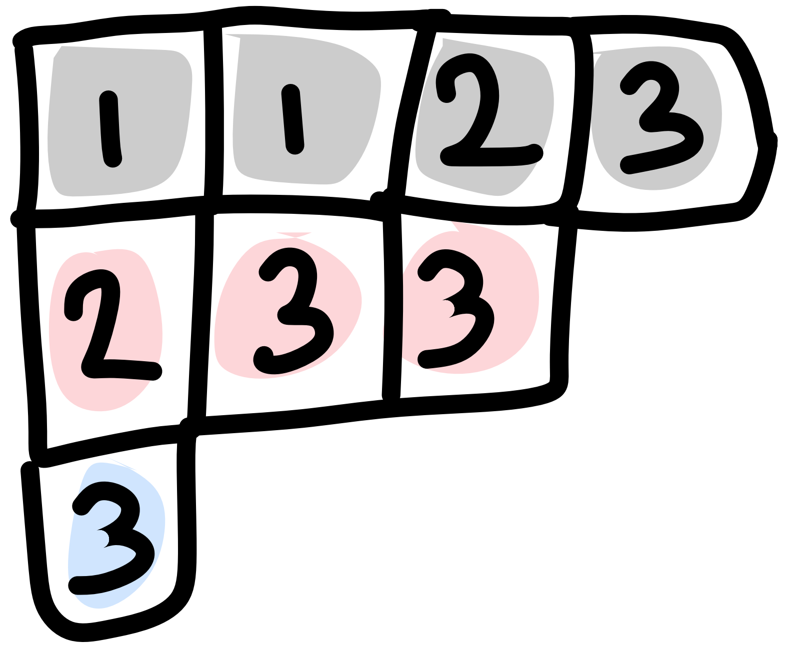

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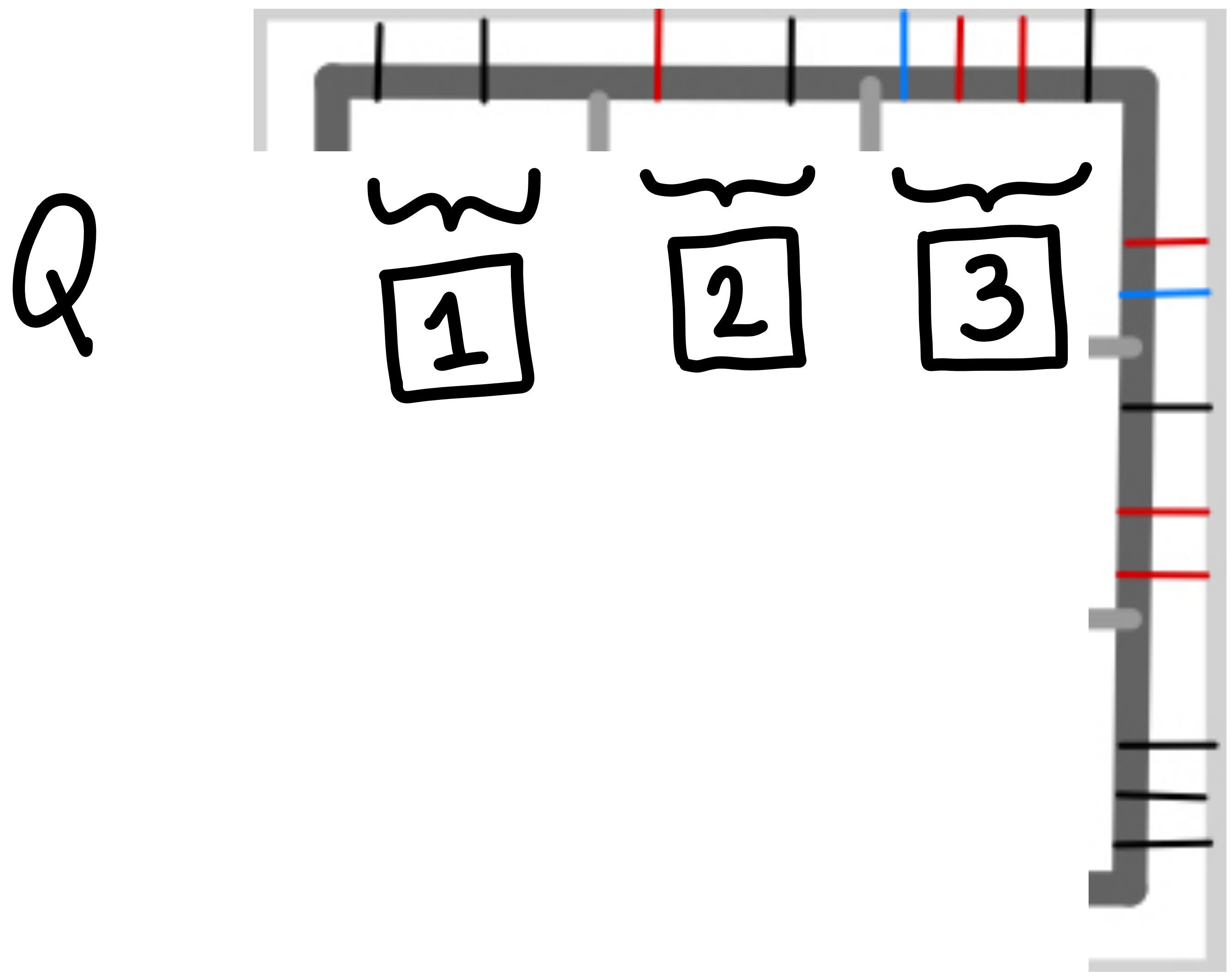

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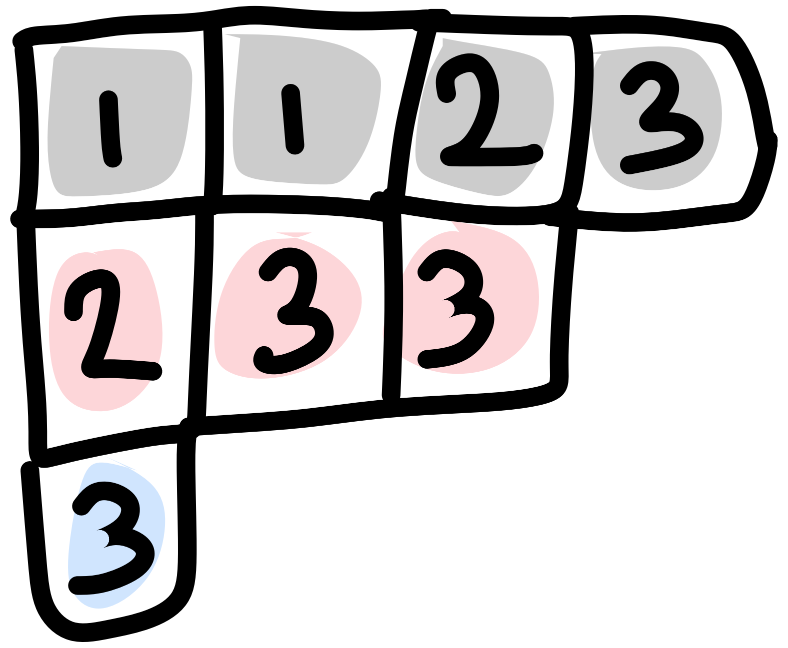

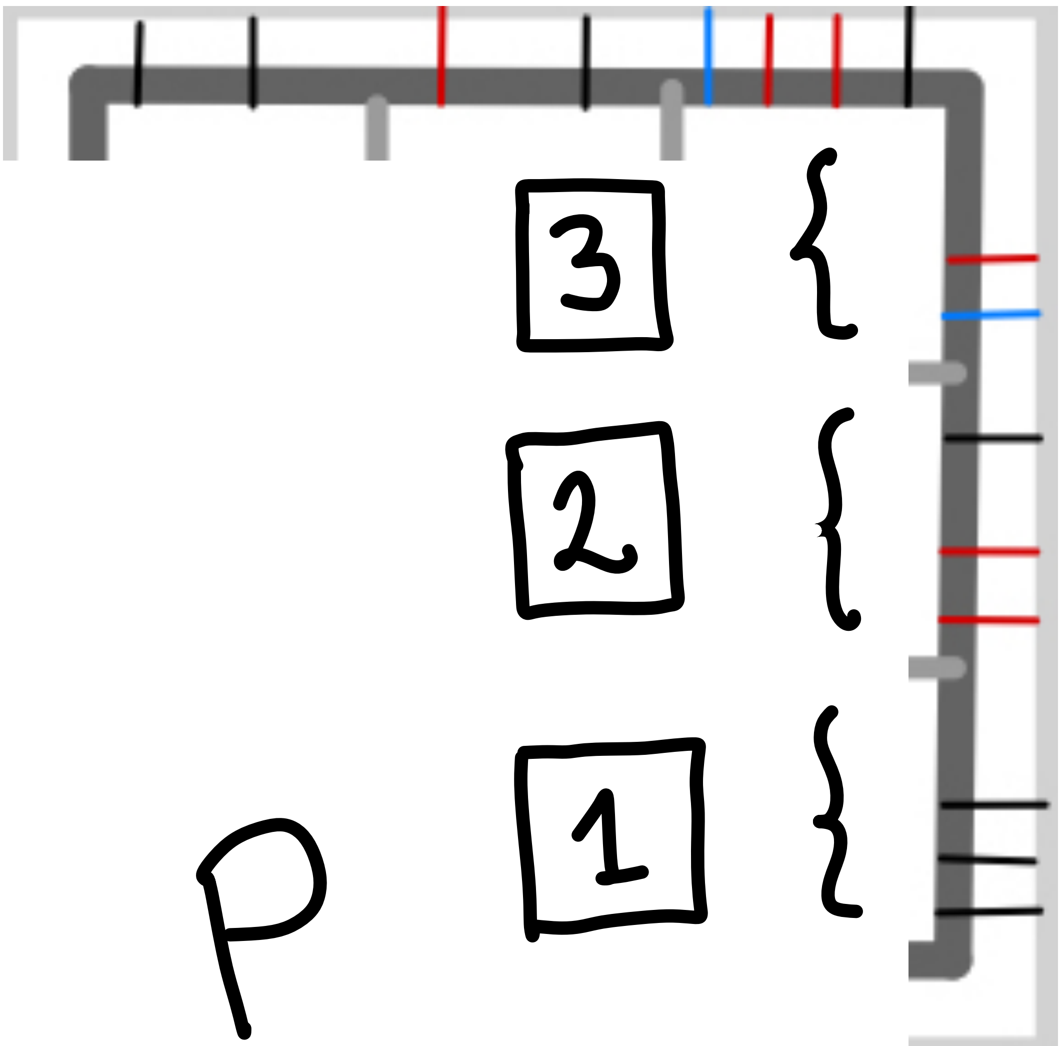

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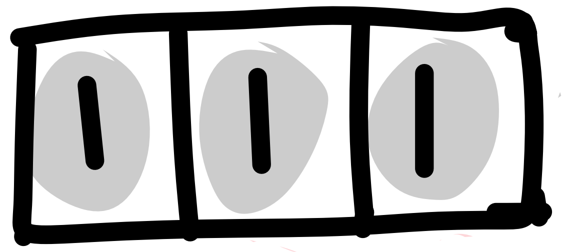

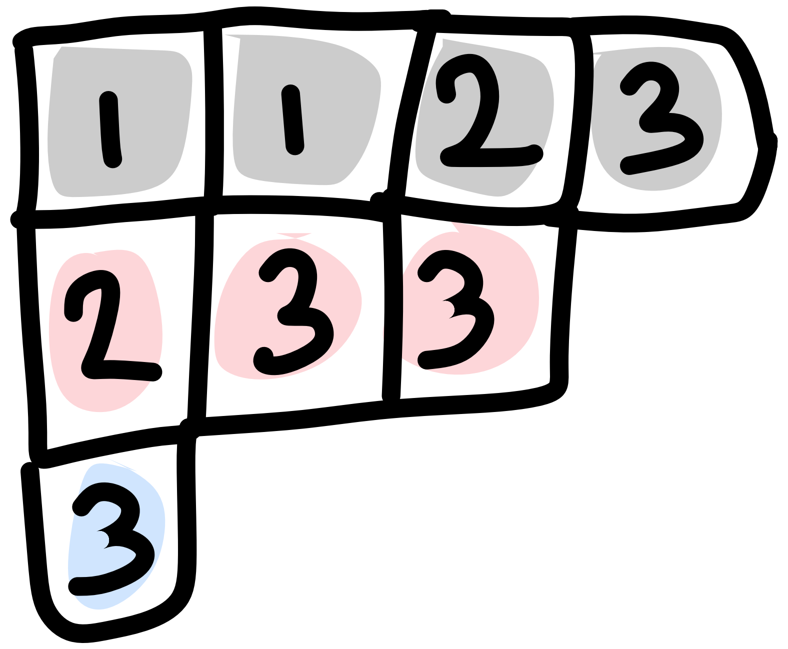

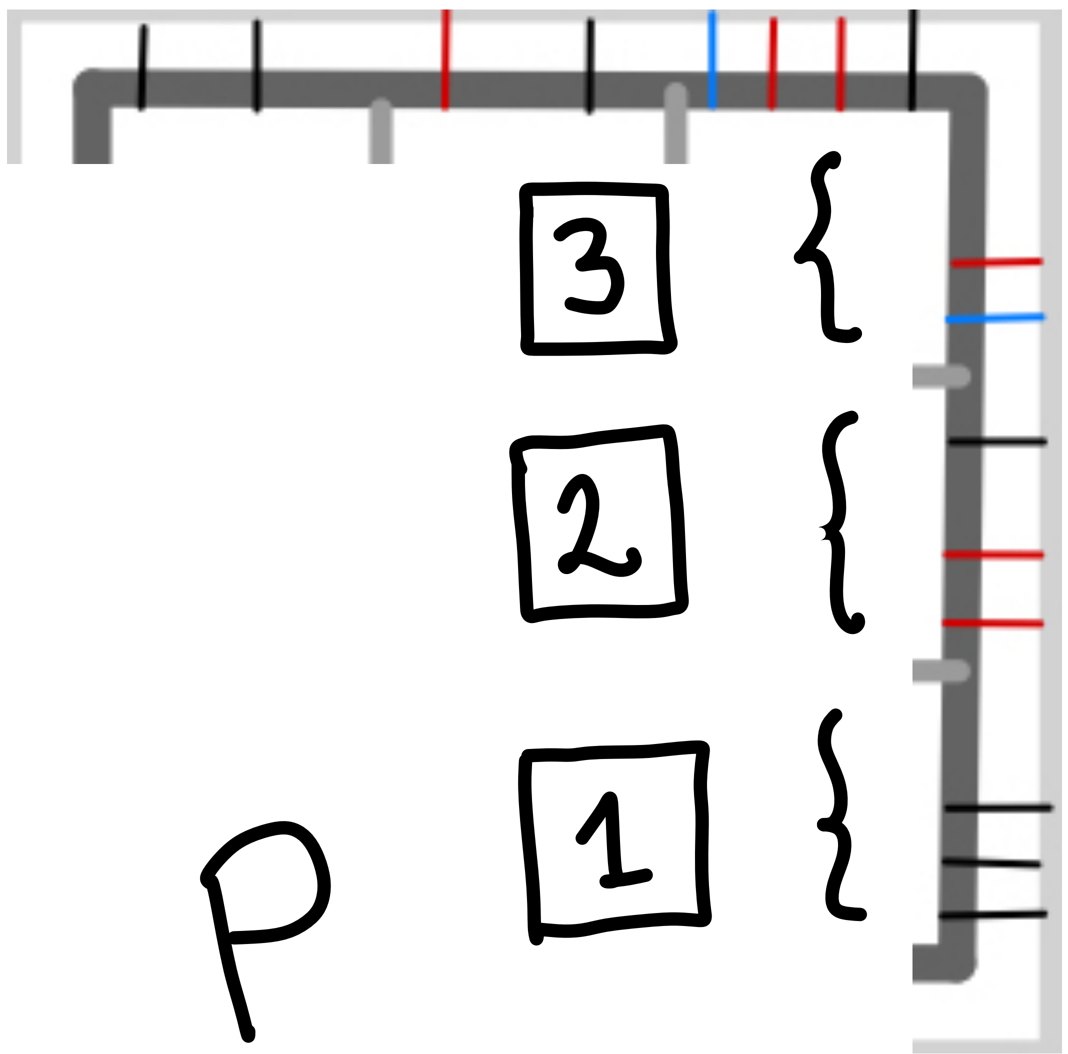

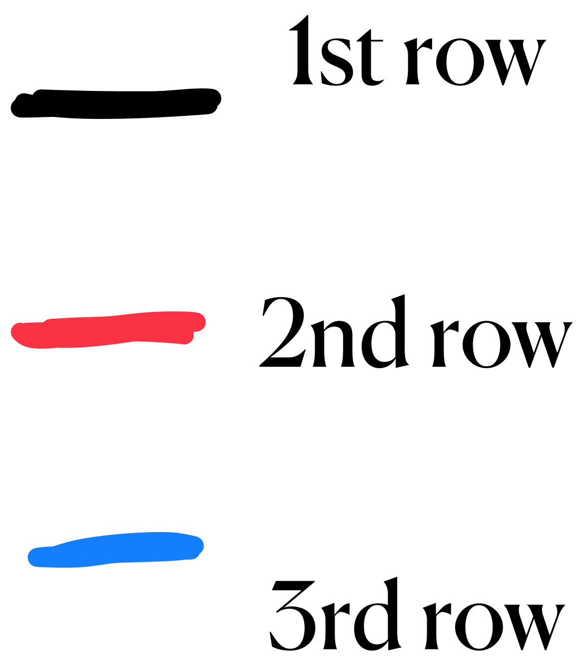

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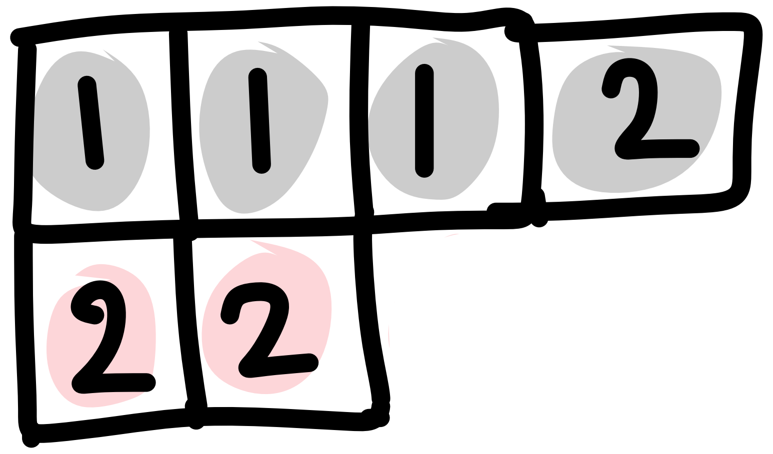

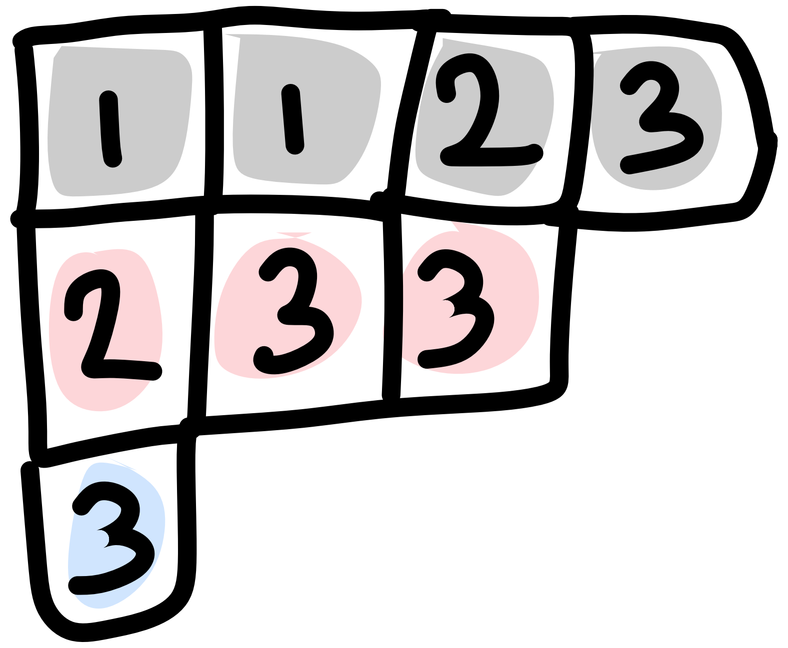

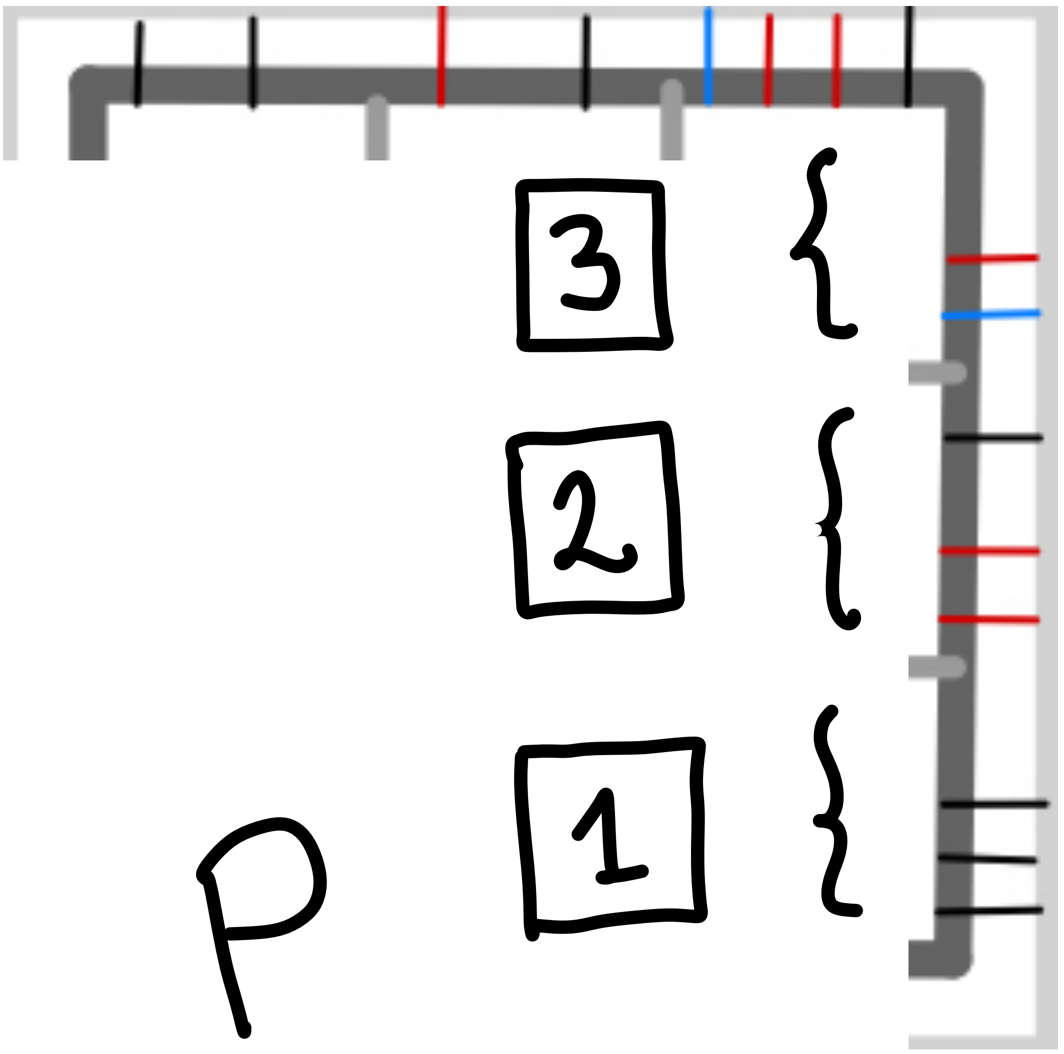

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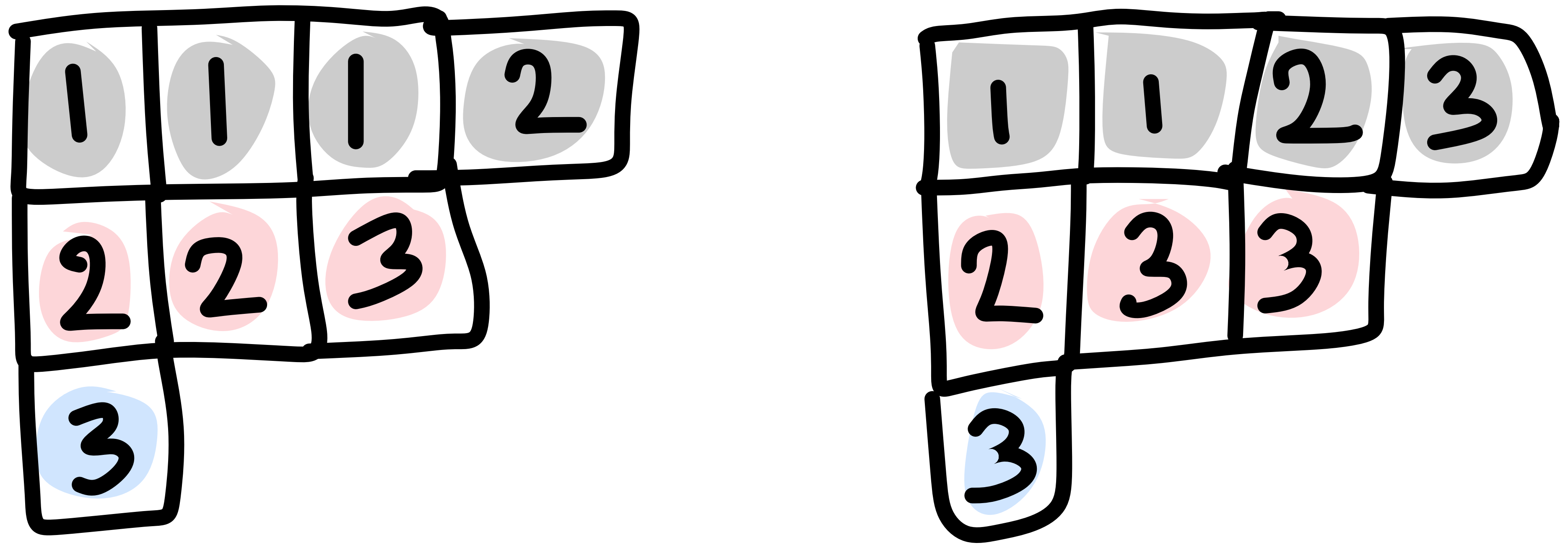

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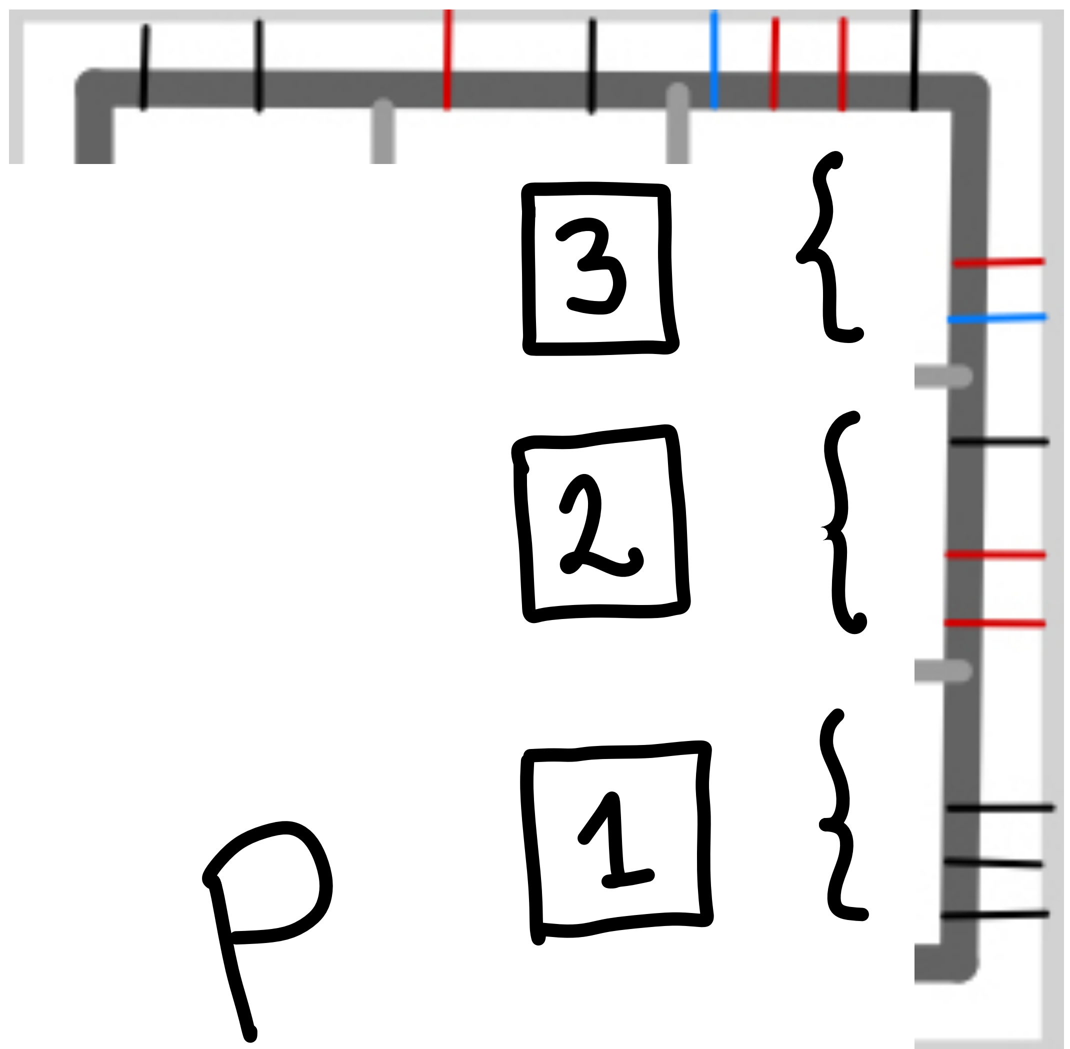

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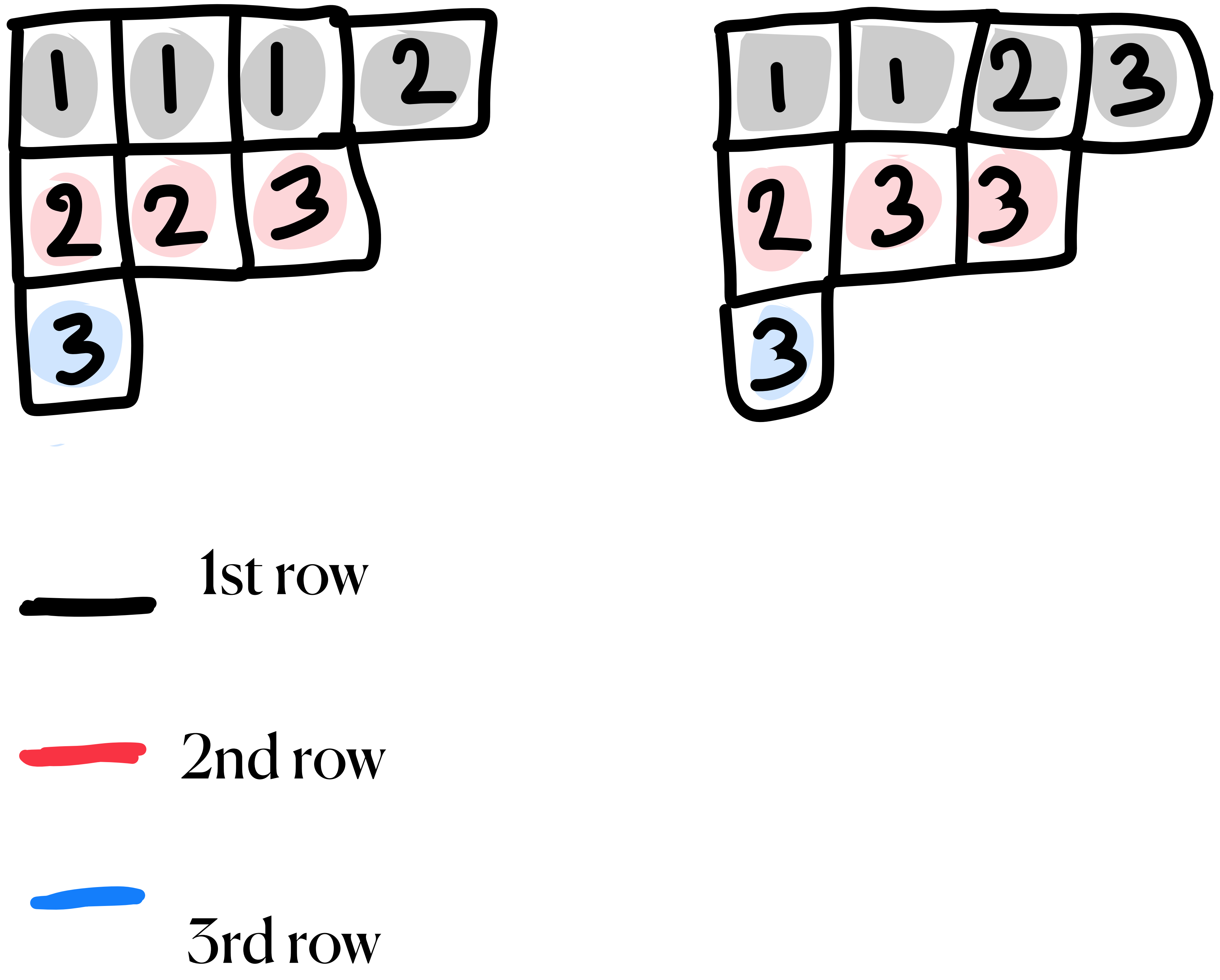

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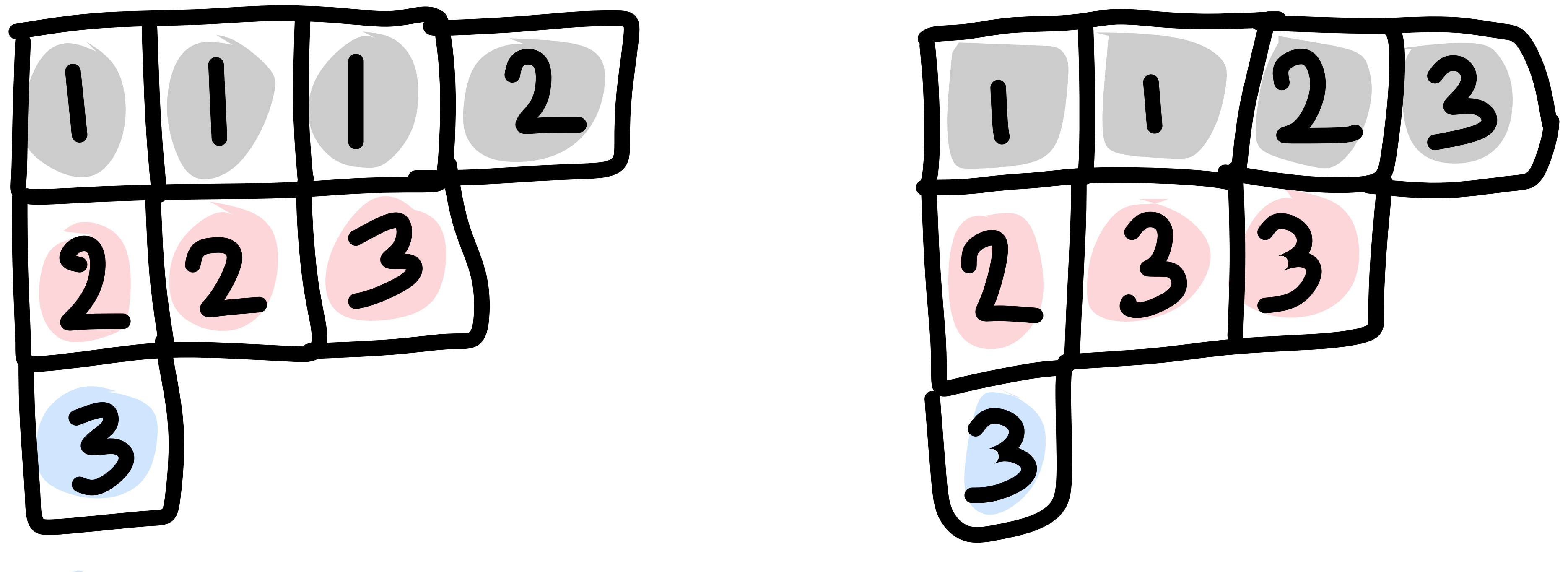

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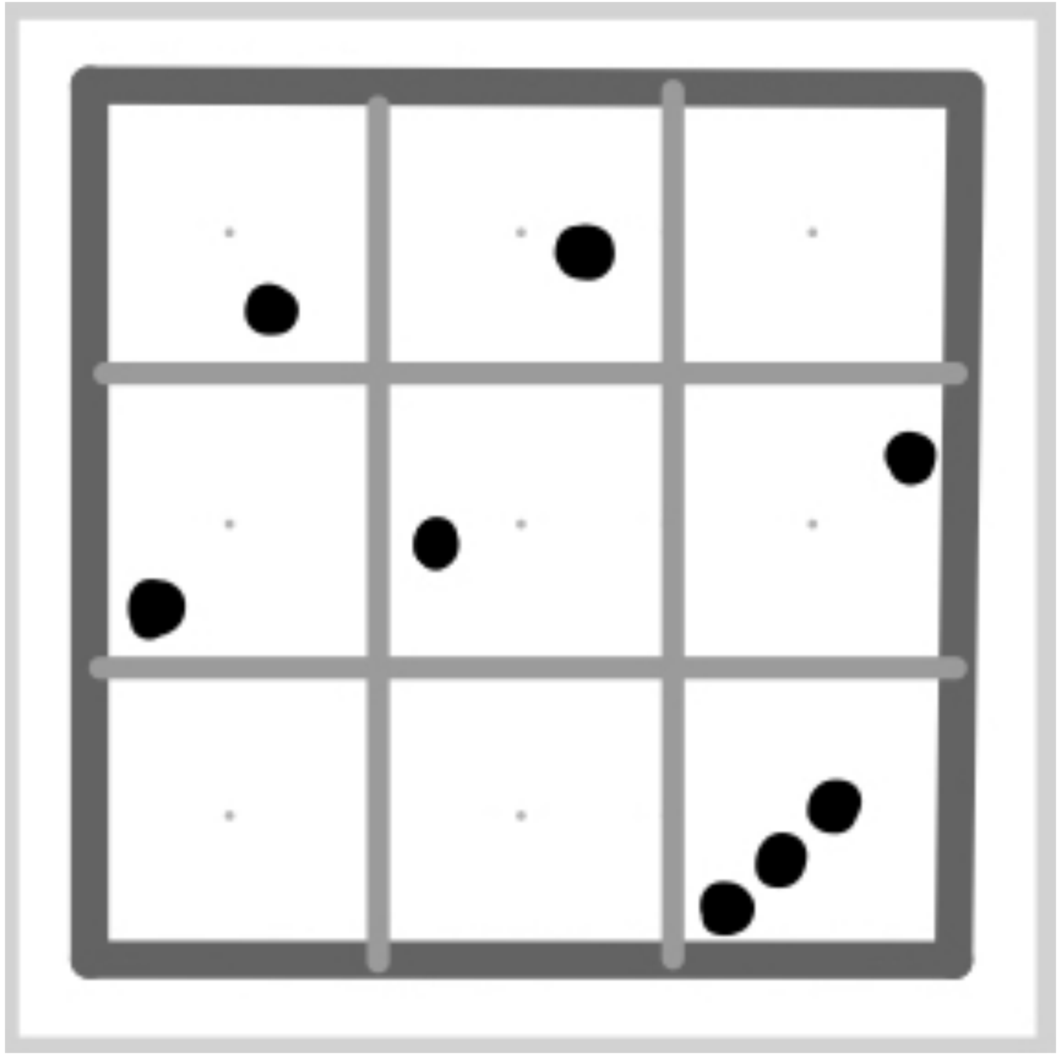
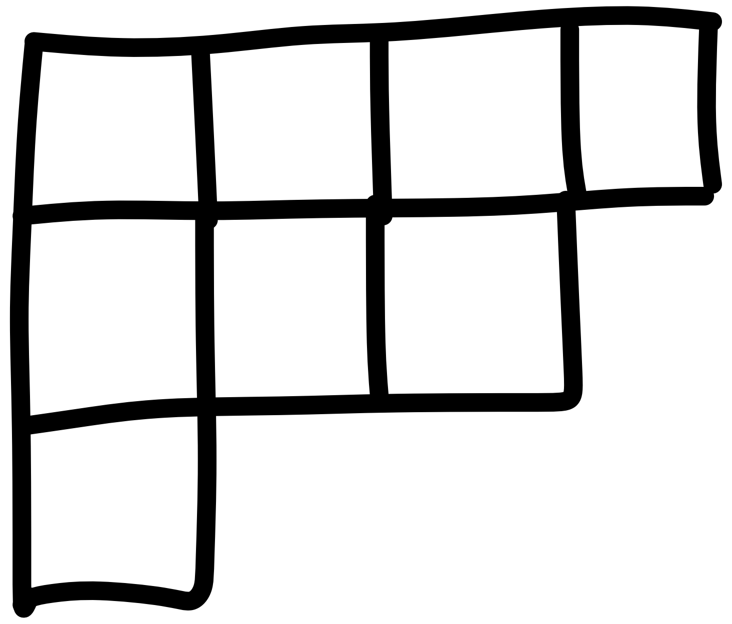


$\lambda =$ 

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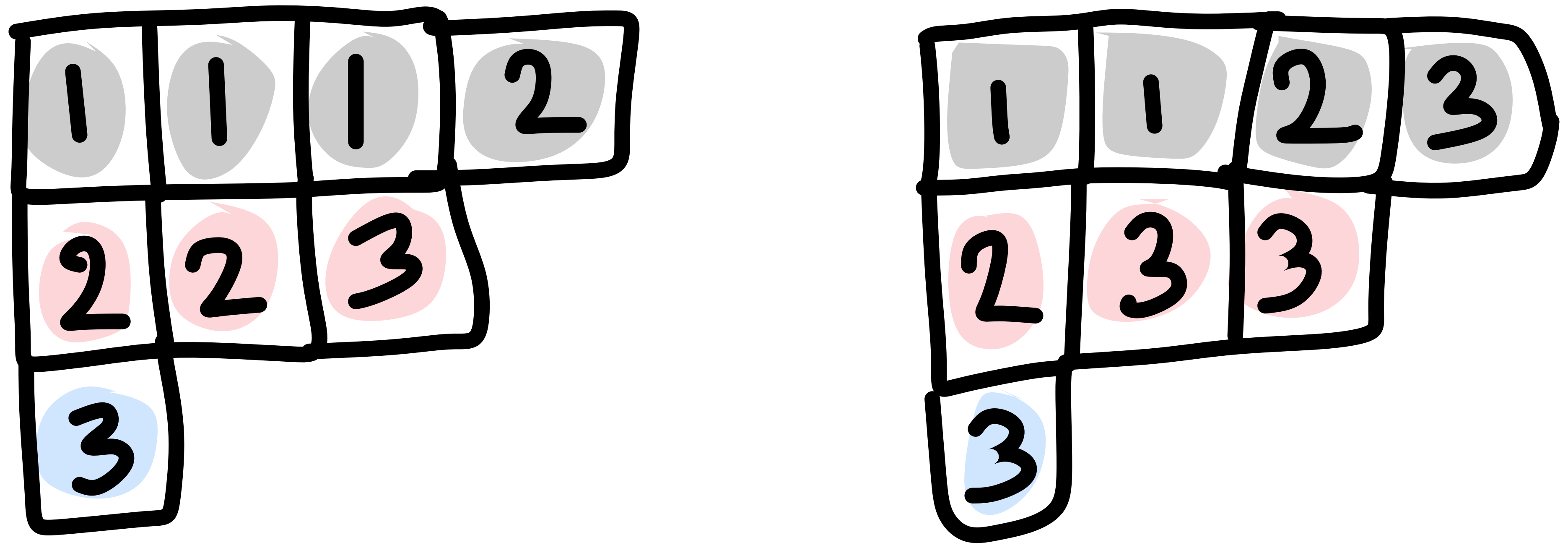
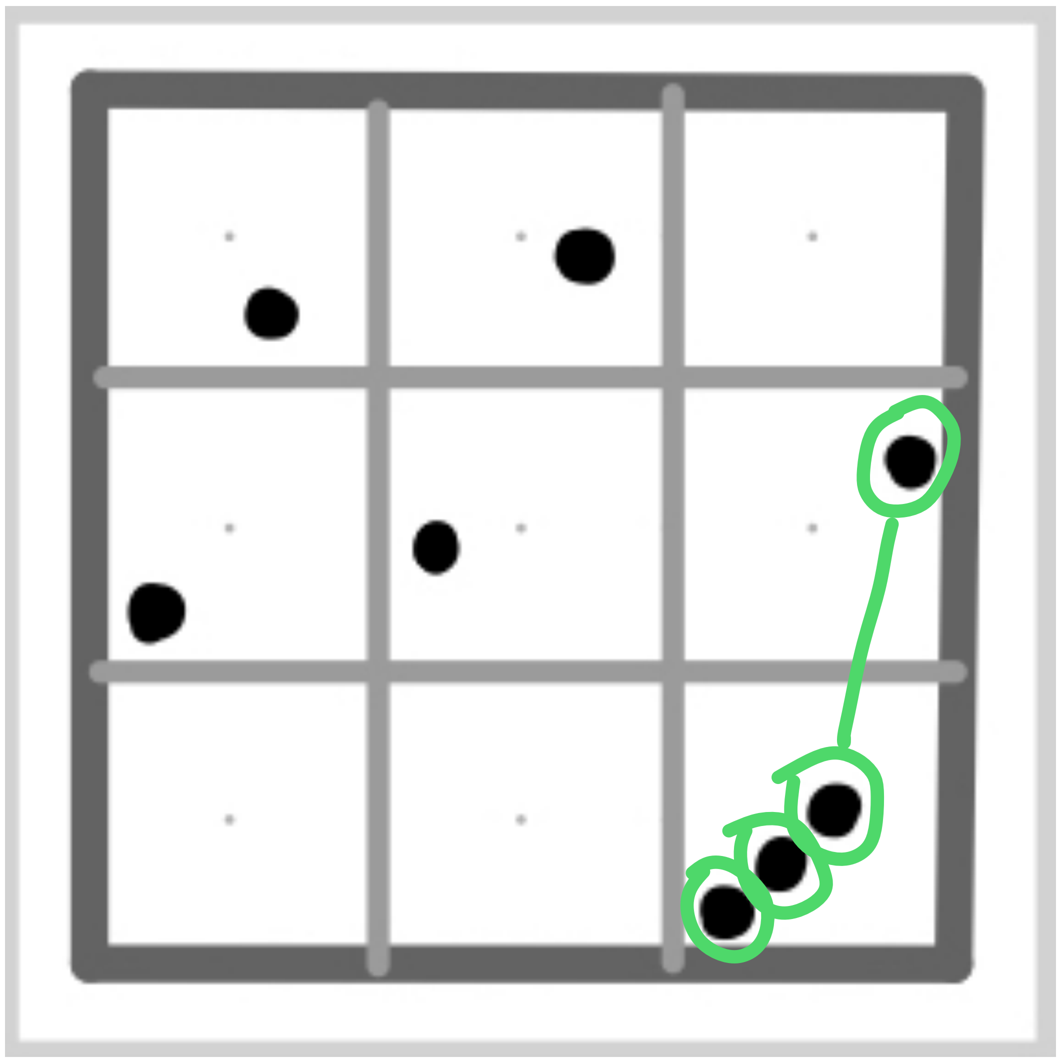


Example



$$\lambda_1 = \mathrm{LIS}_1$$

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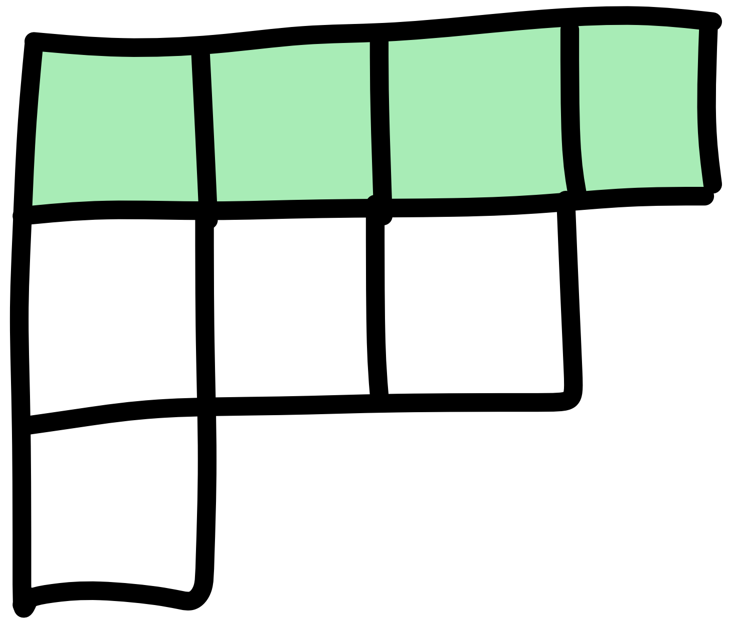

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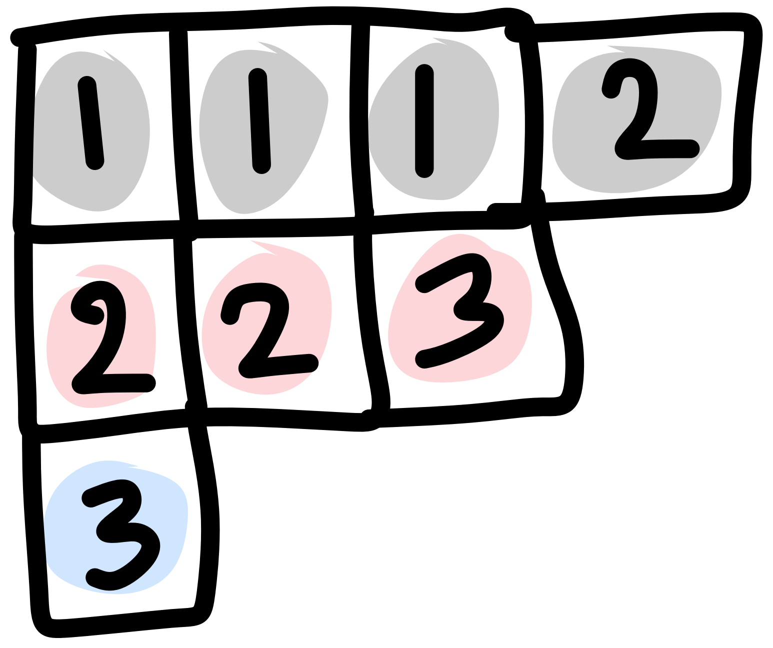

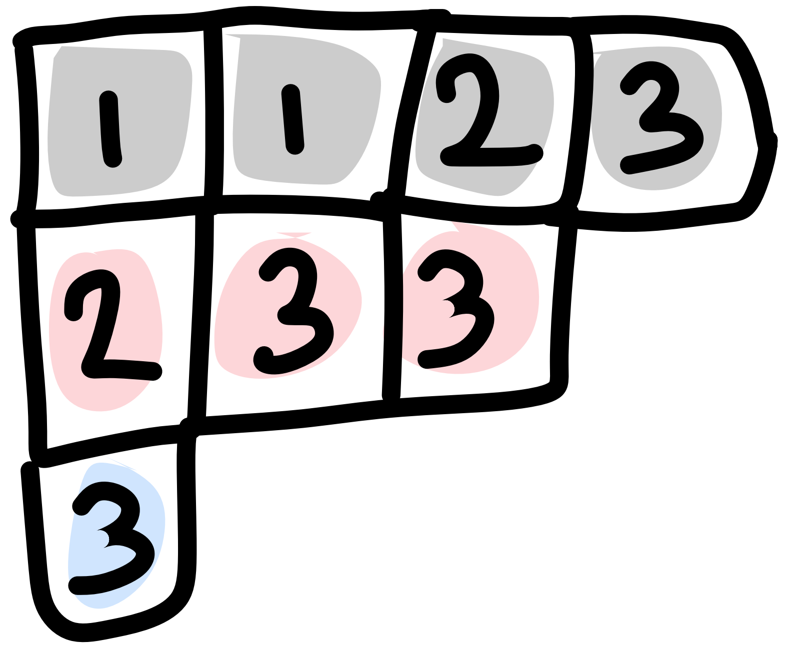

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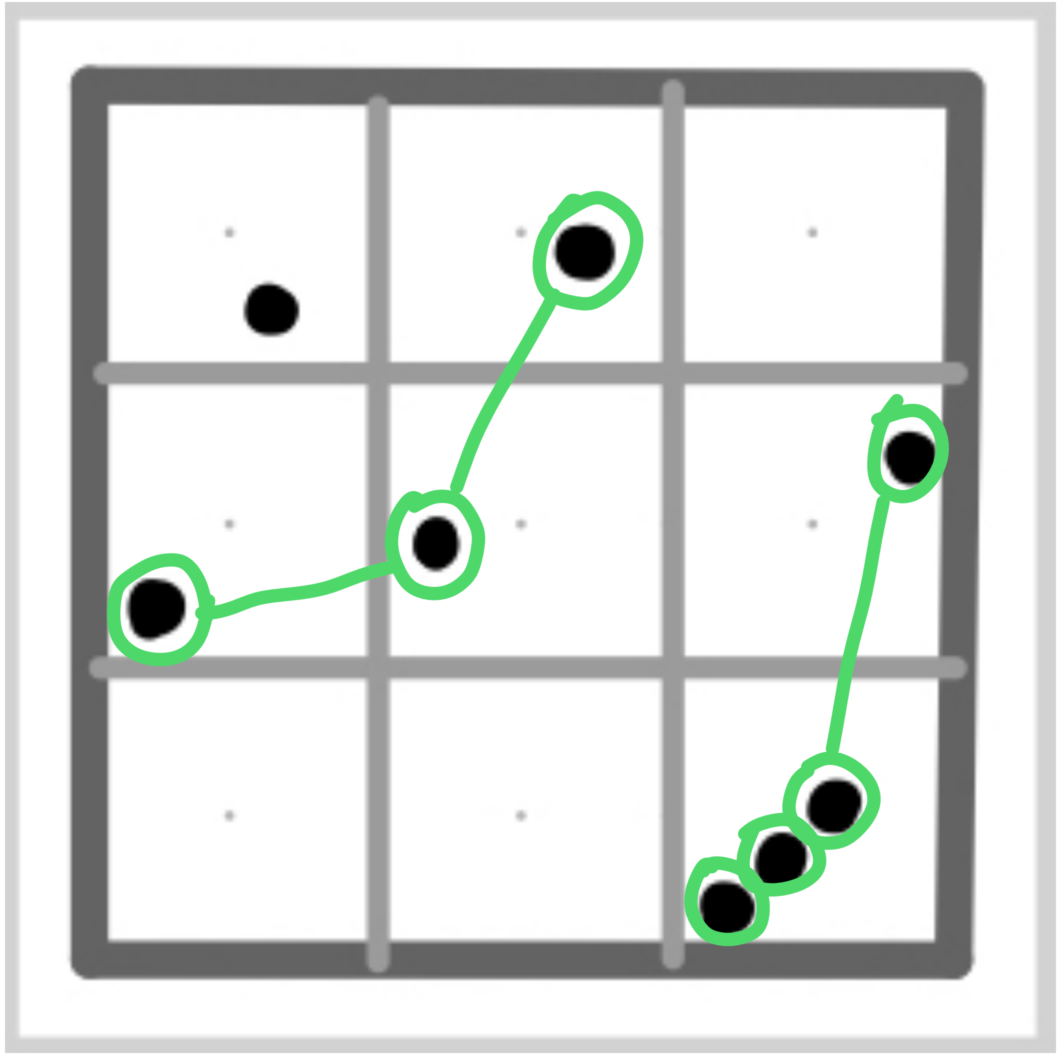
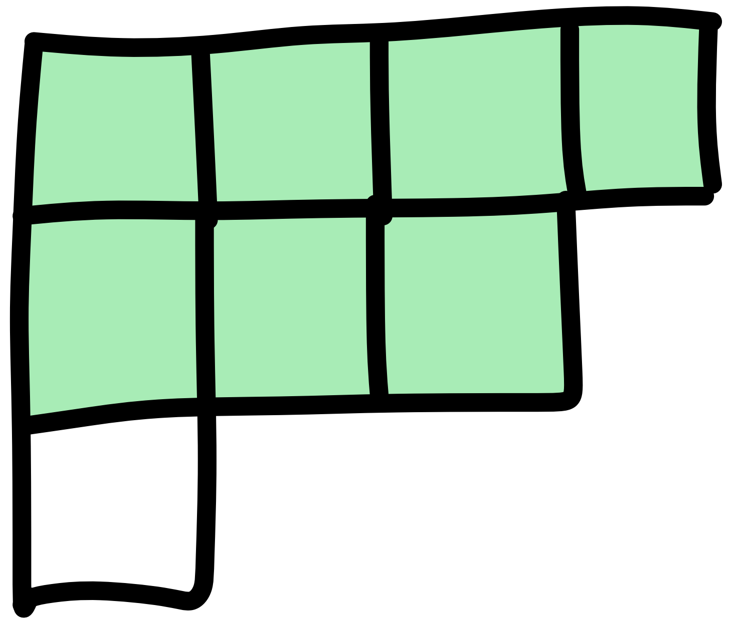


$$\lambda_1 = \mathrm{LIS}_1$$

$$\lambda_1 + \lambda_2 = \mathrm{LIS}_2$$

$$\lambda =$$



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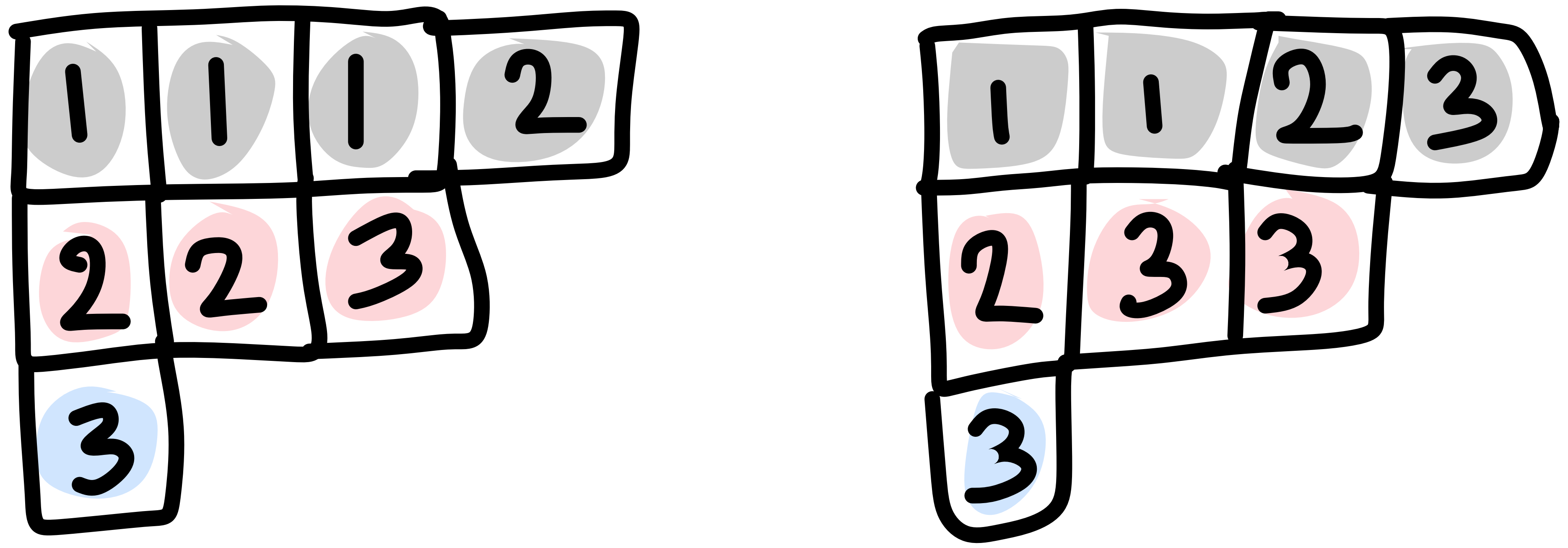
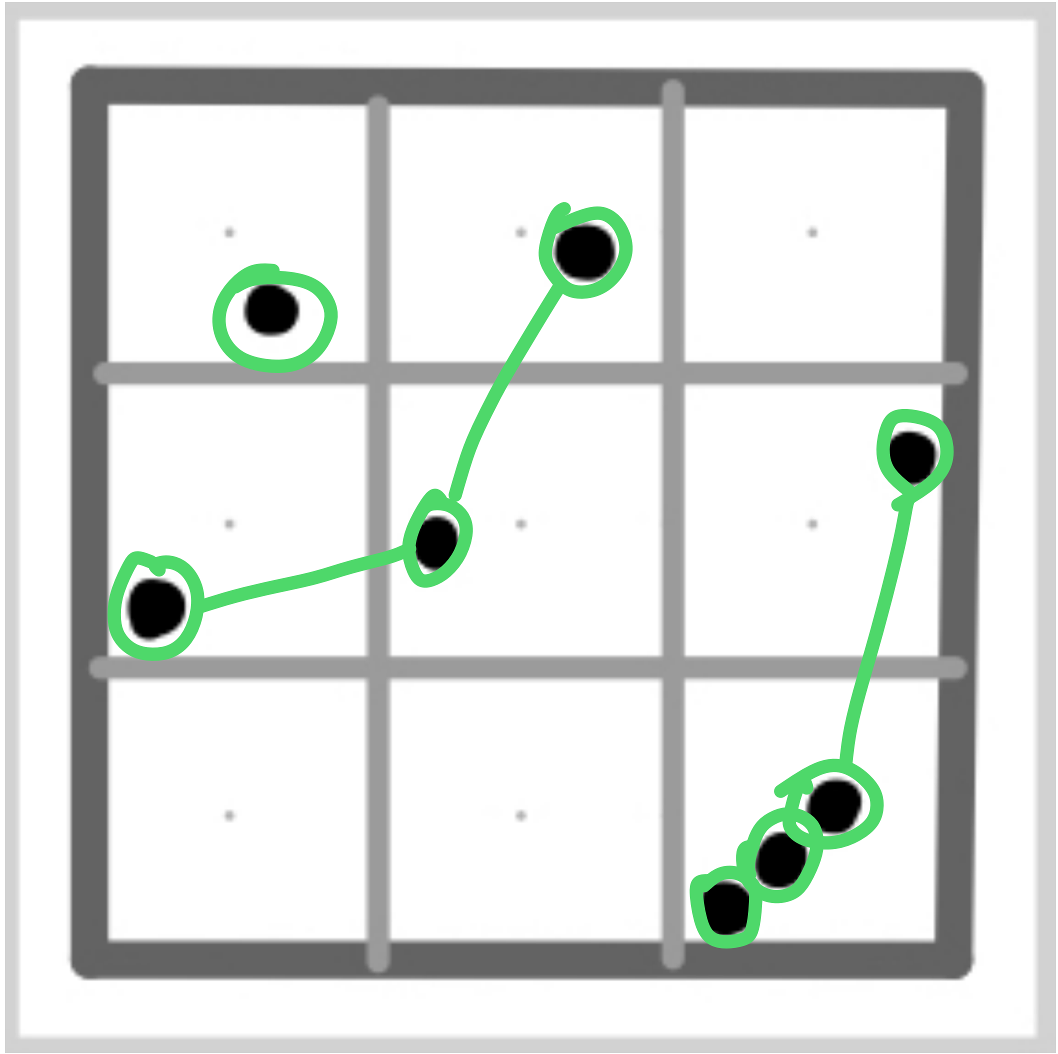
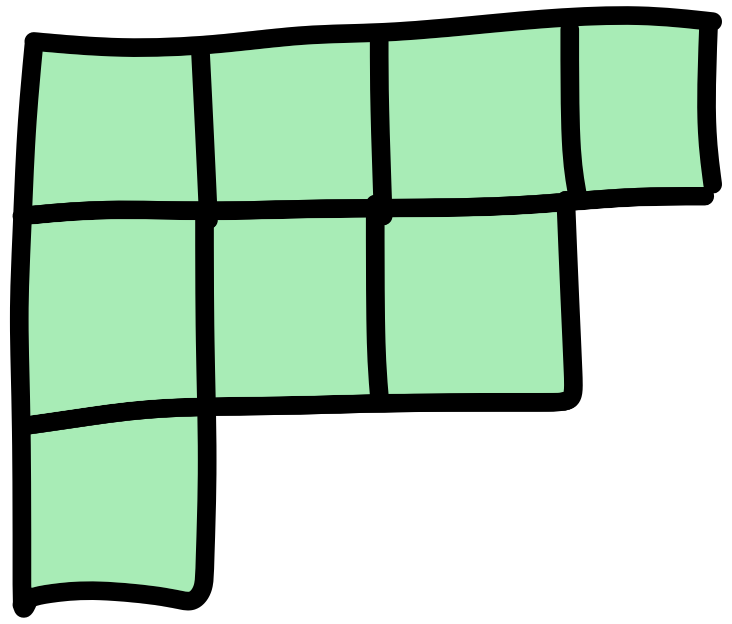




$$\lambda_1 = \mathrm{LIS}_1$$
$$\lambda_1 + \lambda_2 = \mathrm{LIS}_2$$
$$\lambda_1 + \lambda_2 + \lambda_3 = \mathrm{LIS}_3$$

$$\lambda =$$

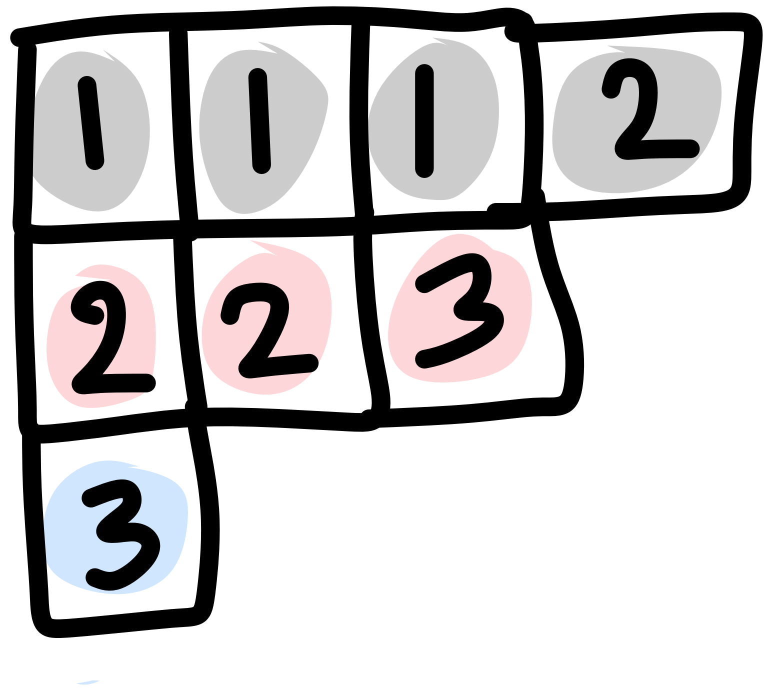


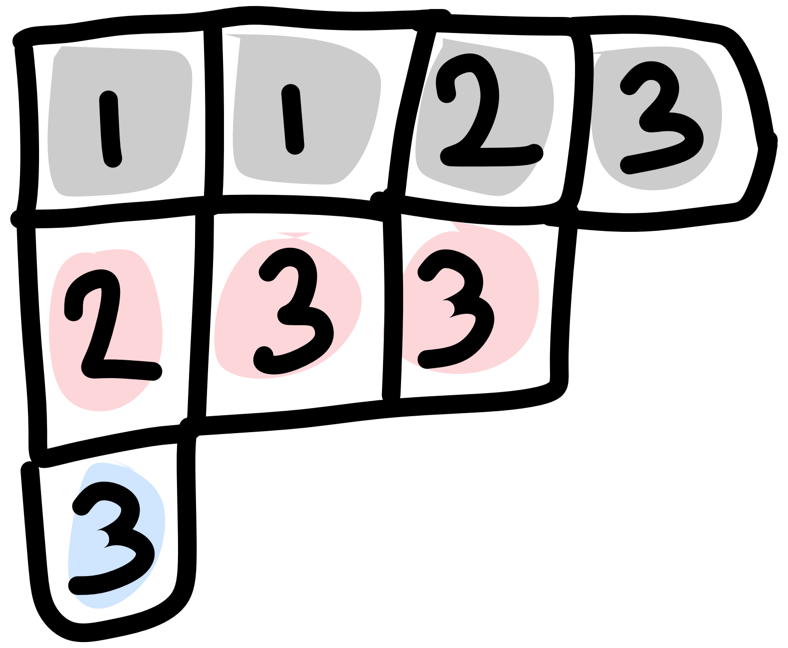
RSK correspondence

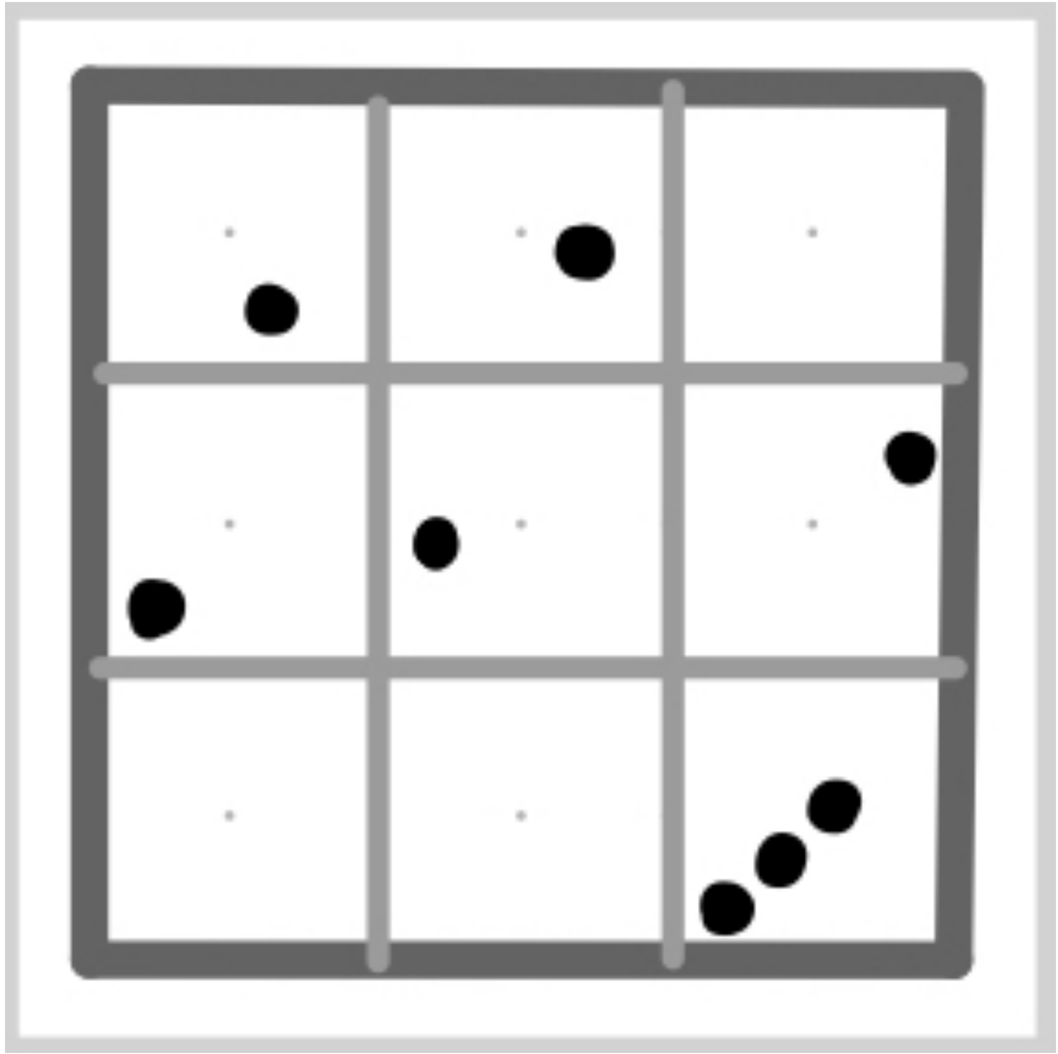
$$\left(M_{i,j}\right)_{i,j=1}^{n} \longleftrightarrow (P, Q)$$

Example

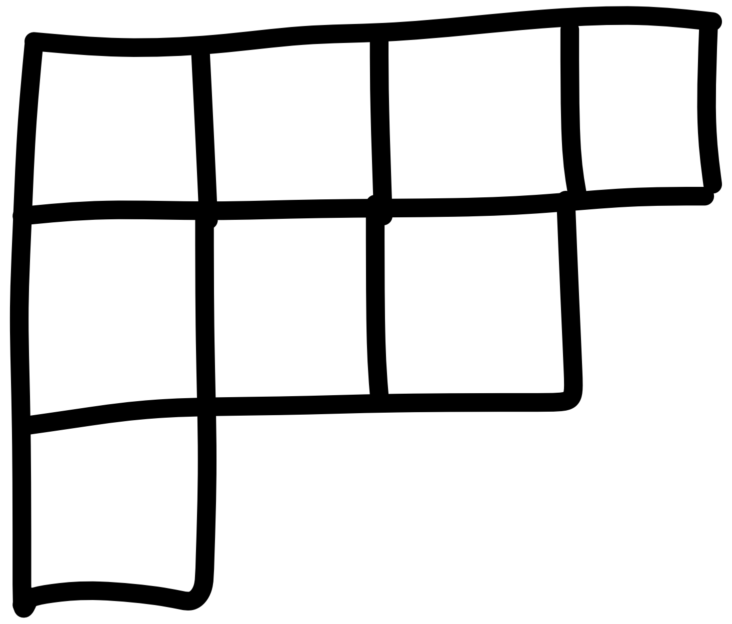
$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix} \longleftrightarrow$$







$\lambda =$



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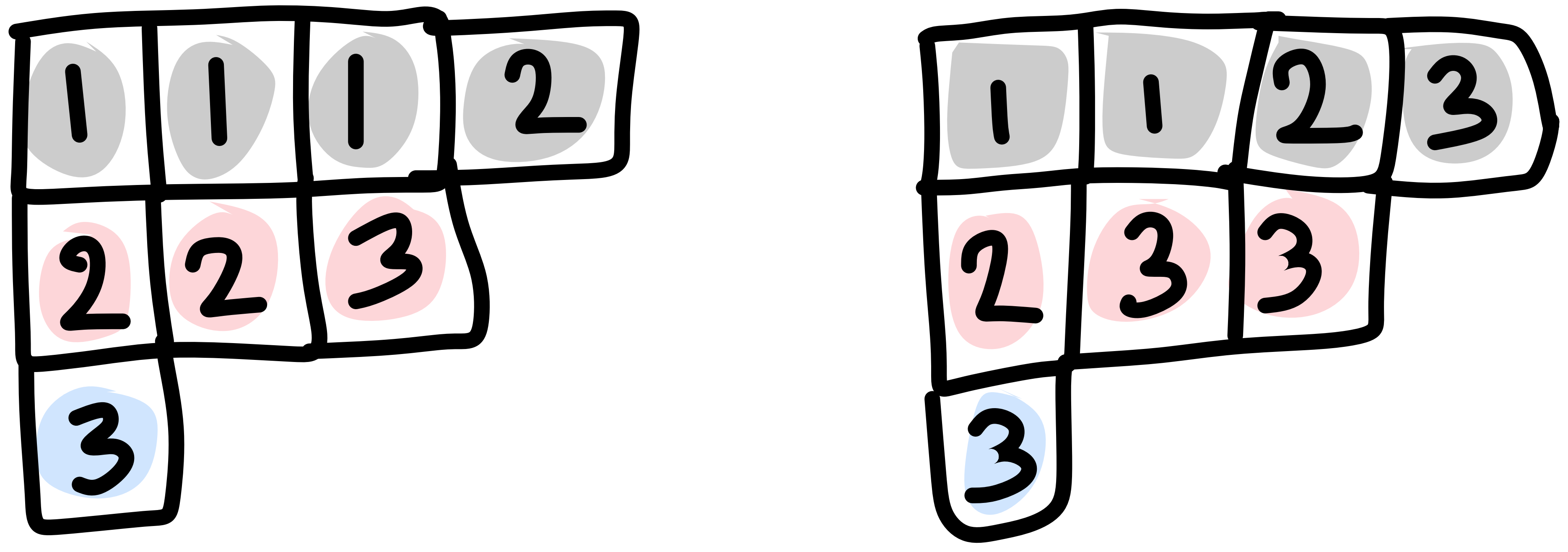
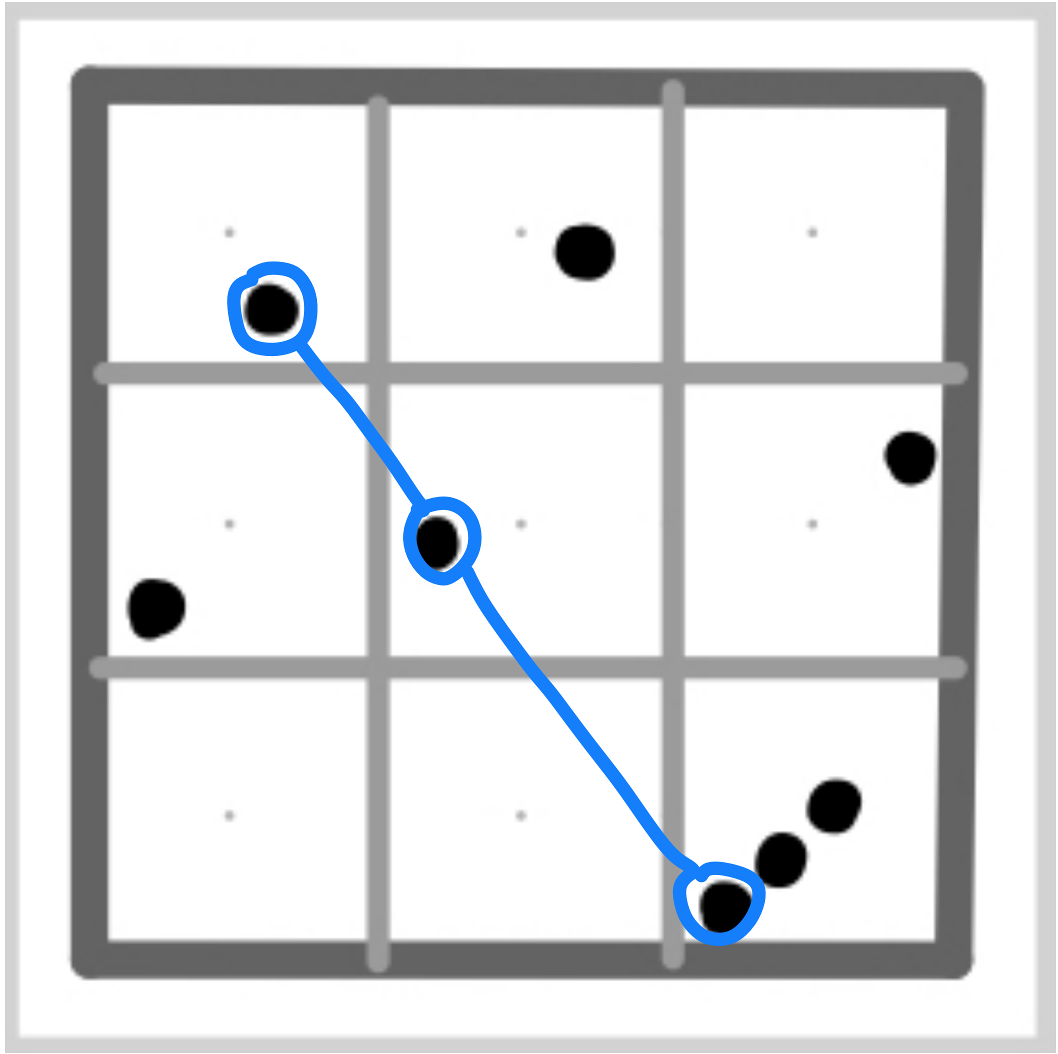
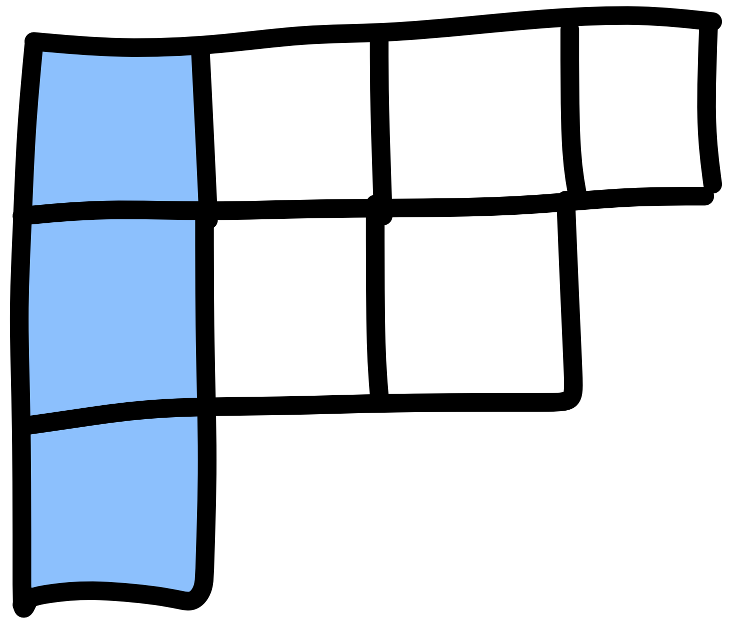




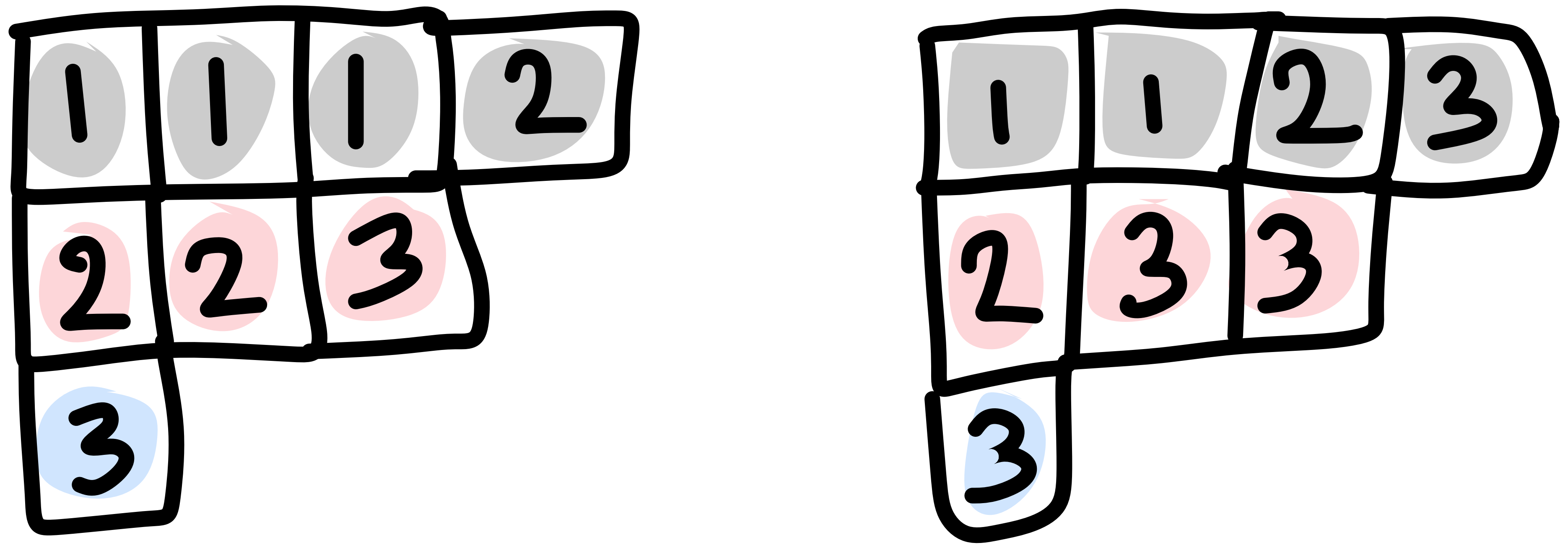
$$\lambda'_1 = \mathrm{LDS}_1$$

$\lambda =$ 

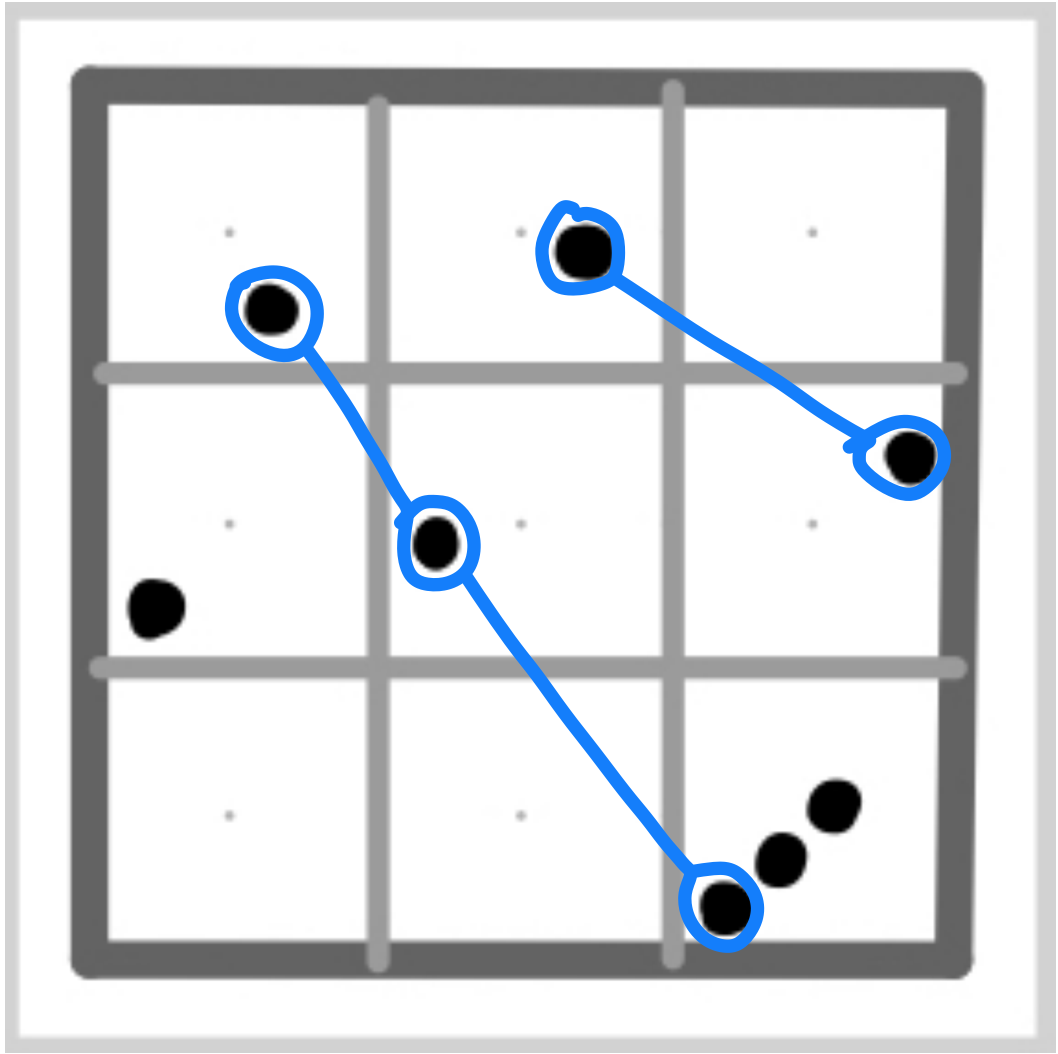
RSK correspondence

$$\left(M_{i,j}\right)_{i,j=1}^{n} \longleftrightarrow (P, Q)$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix} \longleftrightarrow$$



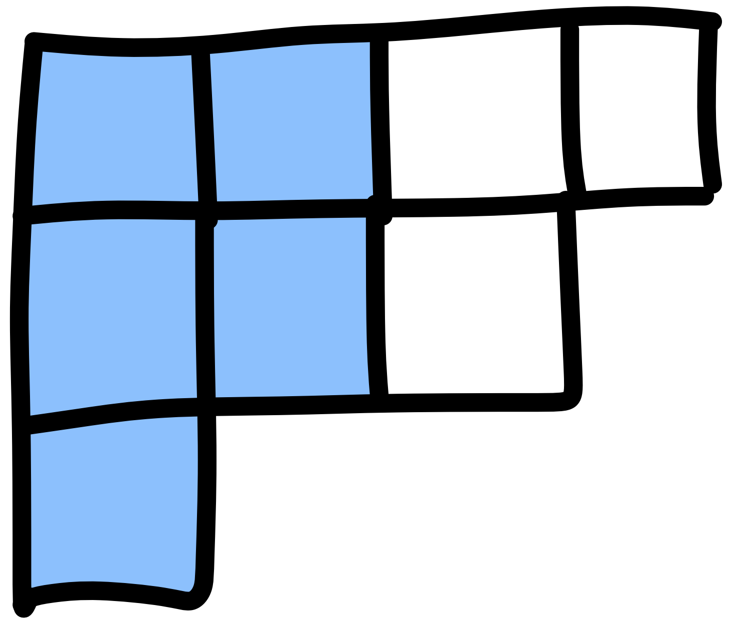
Example



$$\lambda_1' = \mathrm{LDS}_1$$

$$\lambda_1' + \lambda_2' = \mathrm{LDS}_2$$

$\lambda =$



RSK correspondence

$$\left(M_{i,j}\right)_{i,j=1}^{n} \longleftrightarrow (P, Q)$$

Example

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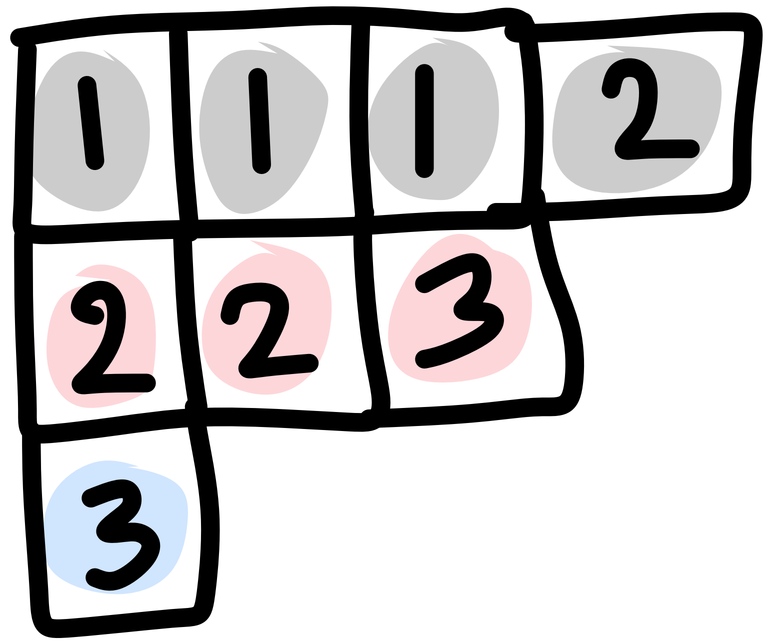

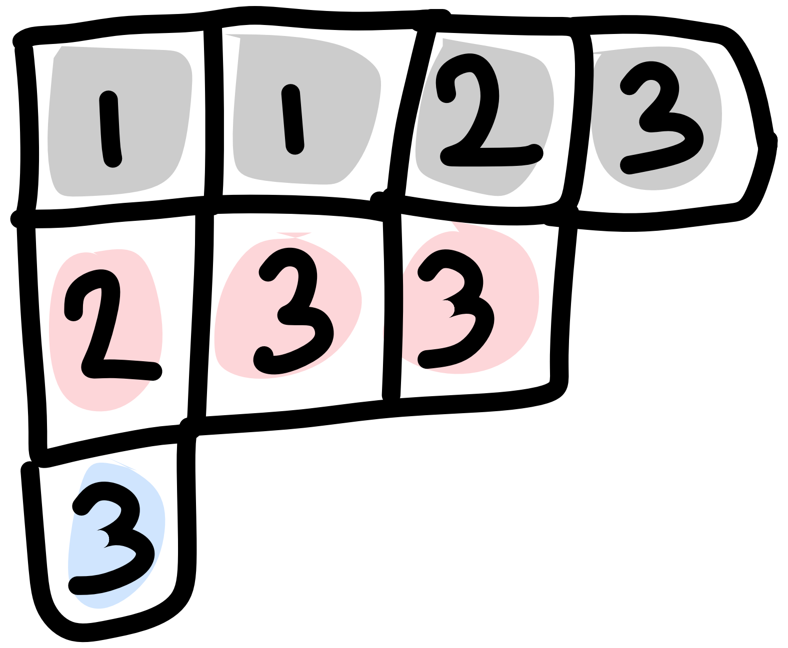

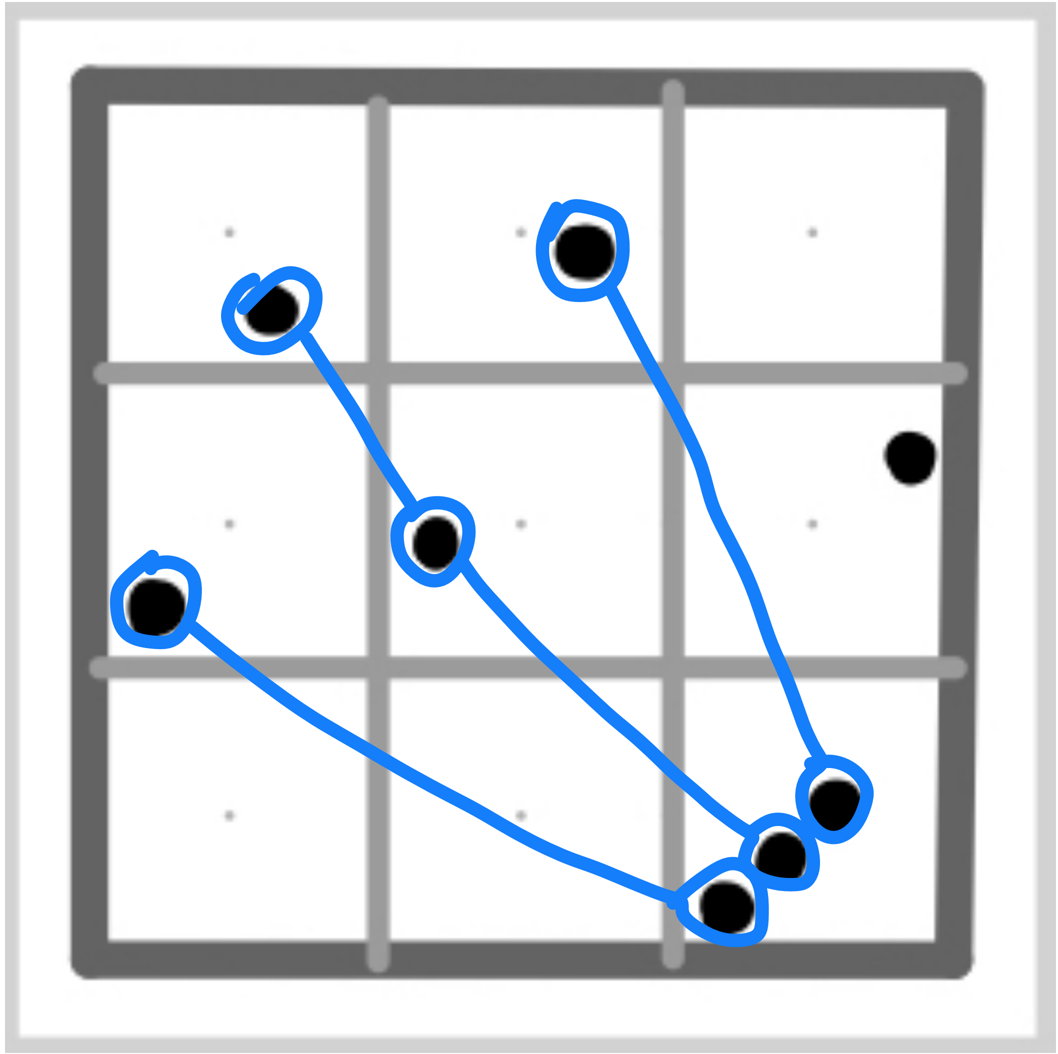

$$\lambda_1' = \mathrm{LDS}_1$$
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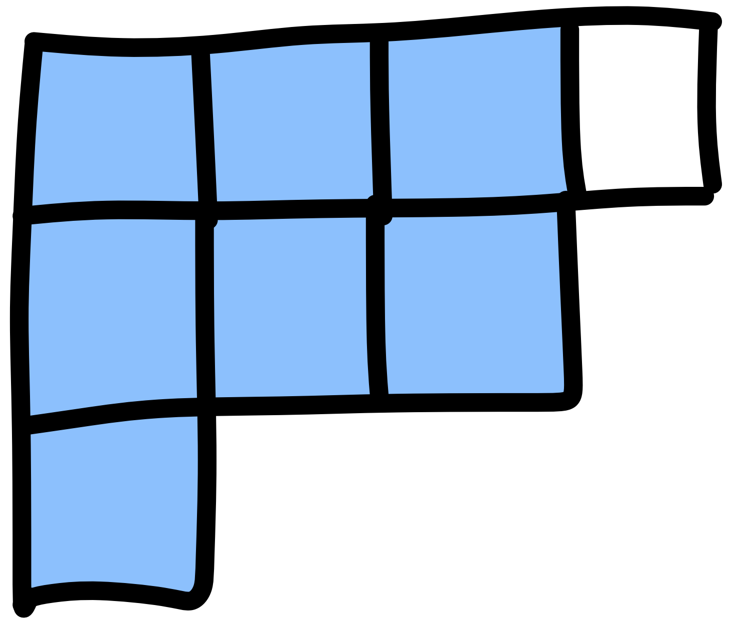
$\lambda =$ 

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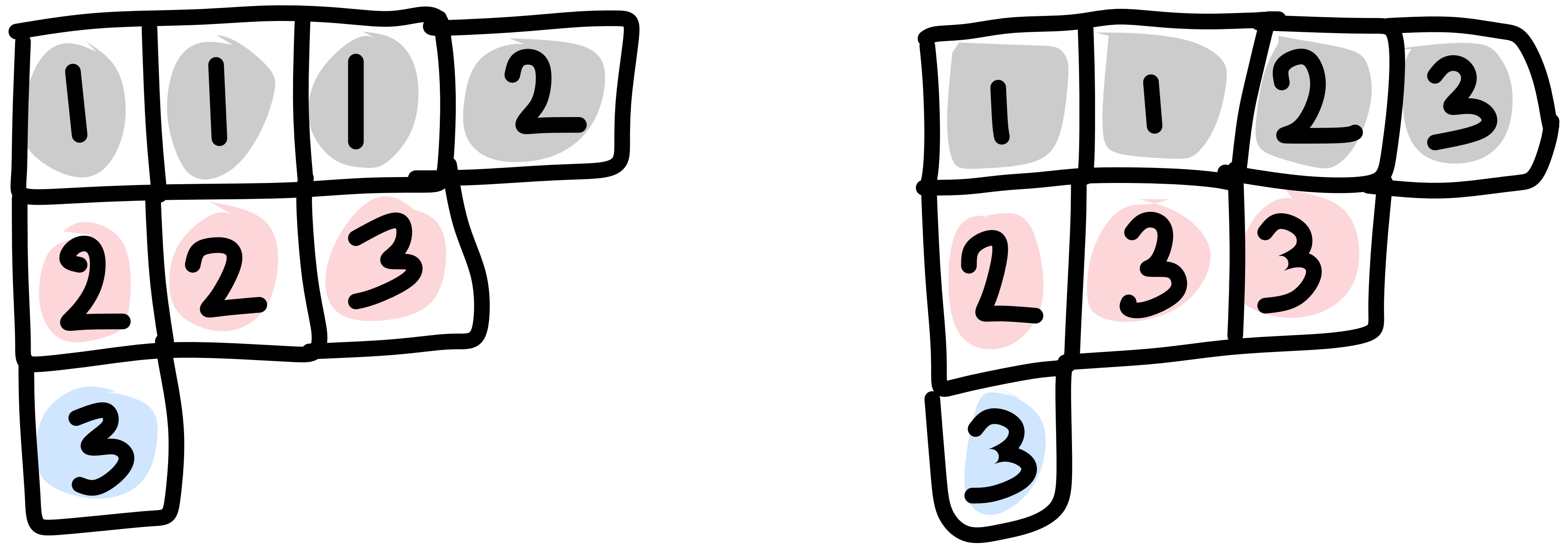
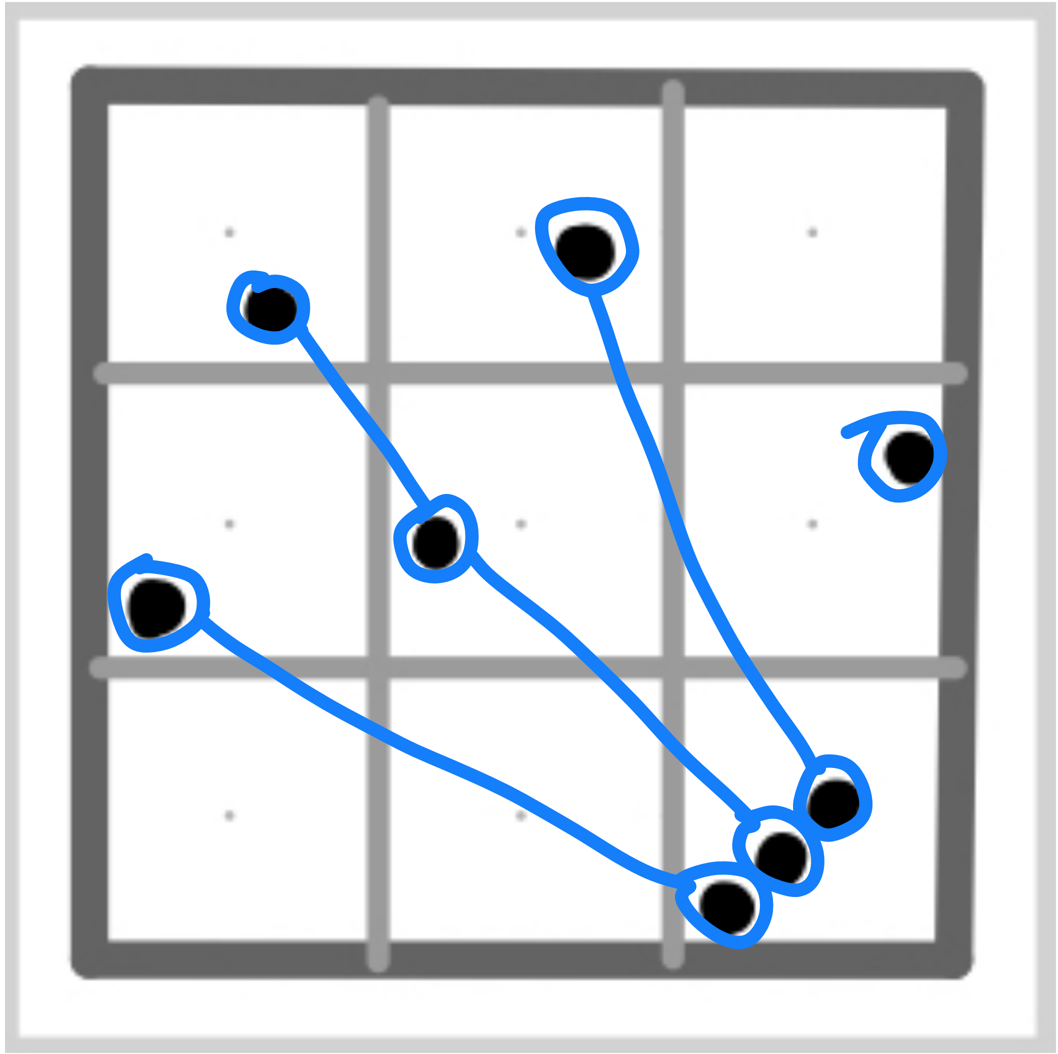


Example



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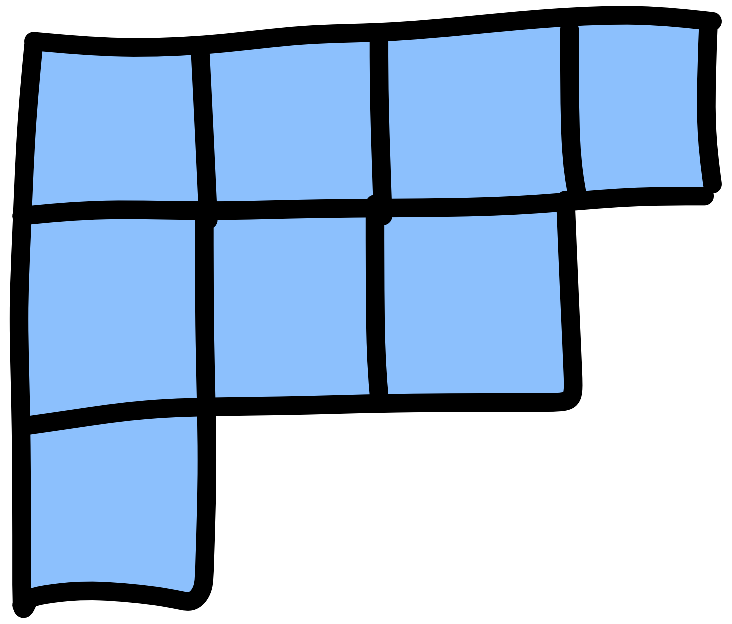
$$\lambda_1' + \lambda_2' + \lambda_3' + \lambda_4' = \mathrm{LDS}_4$$

$\lambda =$ 

Cauchy Identity for *q*-Whittaker polynomials

Cauchy Identity for *q*-Whittaker polynomials

$$\sum_{\mu} b_{\mu}(q) P_{\mu}(x;q) P_{\mu}(y;q) = \prod_{k\geq 0} \prod_{i,j=1}^{n} \frac{1}{1 - x_i y_j q^k}$$

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$$P_\mu(x;q) = \sum_{V \in VST(\mu)} q^{\mathscr{H}(V)} x^V$$

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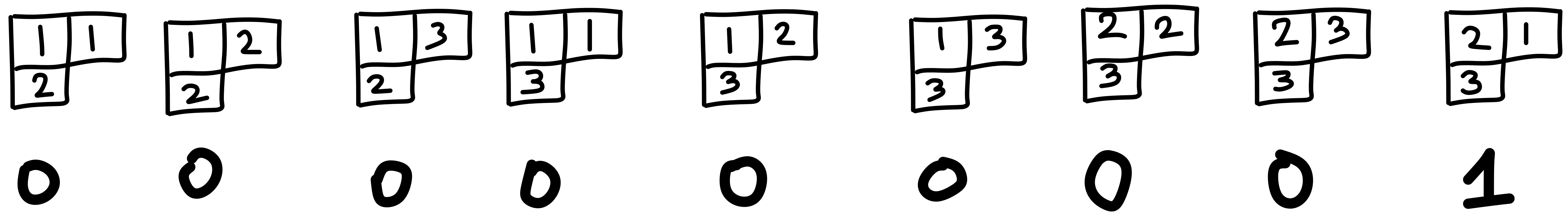
$\mathscr{H}$ = intrinsic energy (almost defined in Lecouvey's lecture)



$$\mathscr{P}_\mu(x;q) = x_1^2x_2 + x_1x_2^2 + x_1x_2x_3 + x_1^2x_3 + x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + qx_1x_2x_3$$

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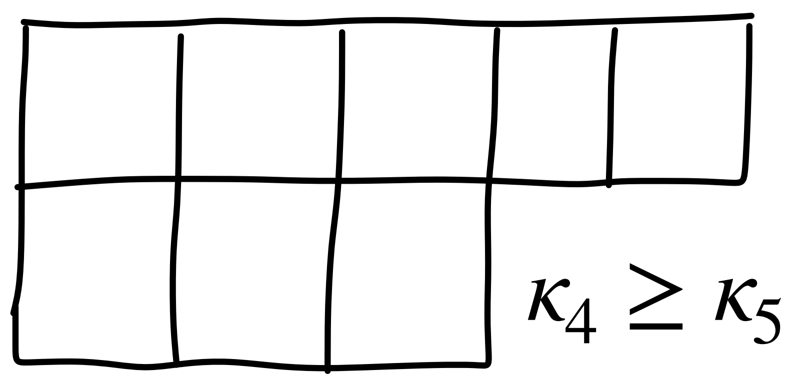
$\mu =$ 

$\kappa_4 \geq \kappa_5$

$\kappa_1 \geq \kappa_2 \geq \kappa_3$

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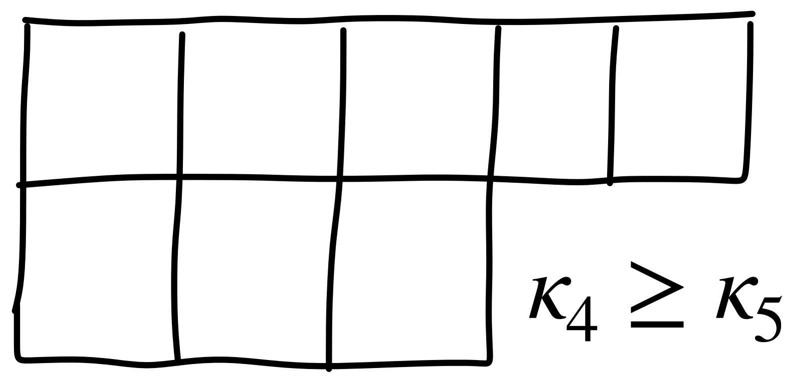
$\mu =$ 

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$$\sum_{\mu} \sum_{\kappa \in \mathscr{K}(\mu)} \sum_{V,W \in VST(\mu)} q^{|\kappa| + \mathscr{H}(V) + \mathscr{H}(W)} x^V y^W = \sum_{M \in \overline{\mathbb{M}}_{n\times n}} \prod_{i,j=1}^{n} (x_i y_j q^k)^{M_{i,j}^k}$$

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Bijective proof: $M \overset{\Upsilon}{\longleftrightarrow} (V, W; \kappa)$: $\sum_{i,j=1}^{n} \sum_{k>0} k M_{i,j}^k = |\kappa| + \mathscr{H}(V) + \mathscr{H}(W)$

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

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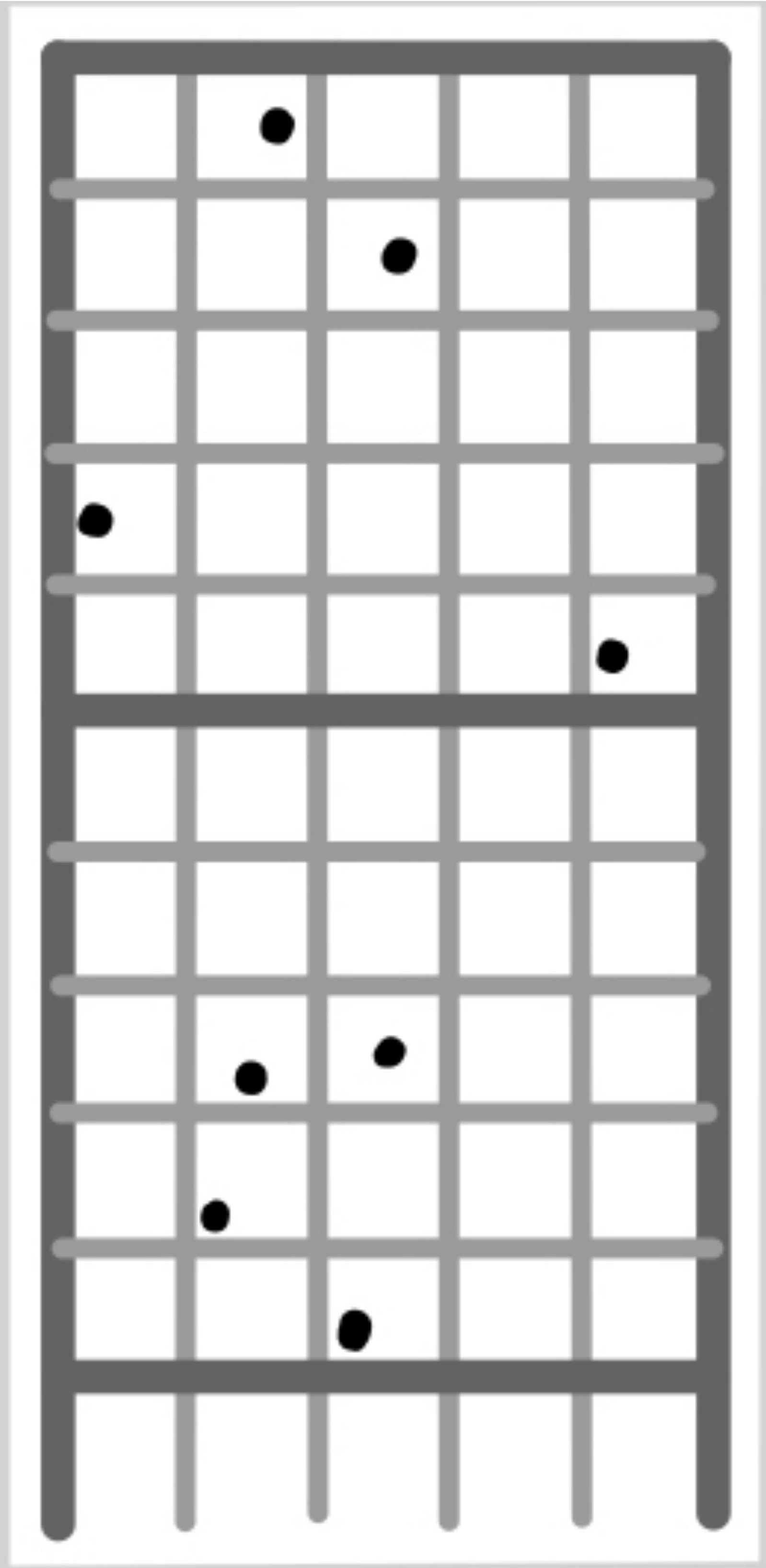
$$\begin{pmatrix} (0,0,\ldots) & (0,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,1,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \end{pmatrix} \longleftrightarrow \; ?$$



Construction of Υ

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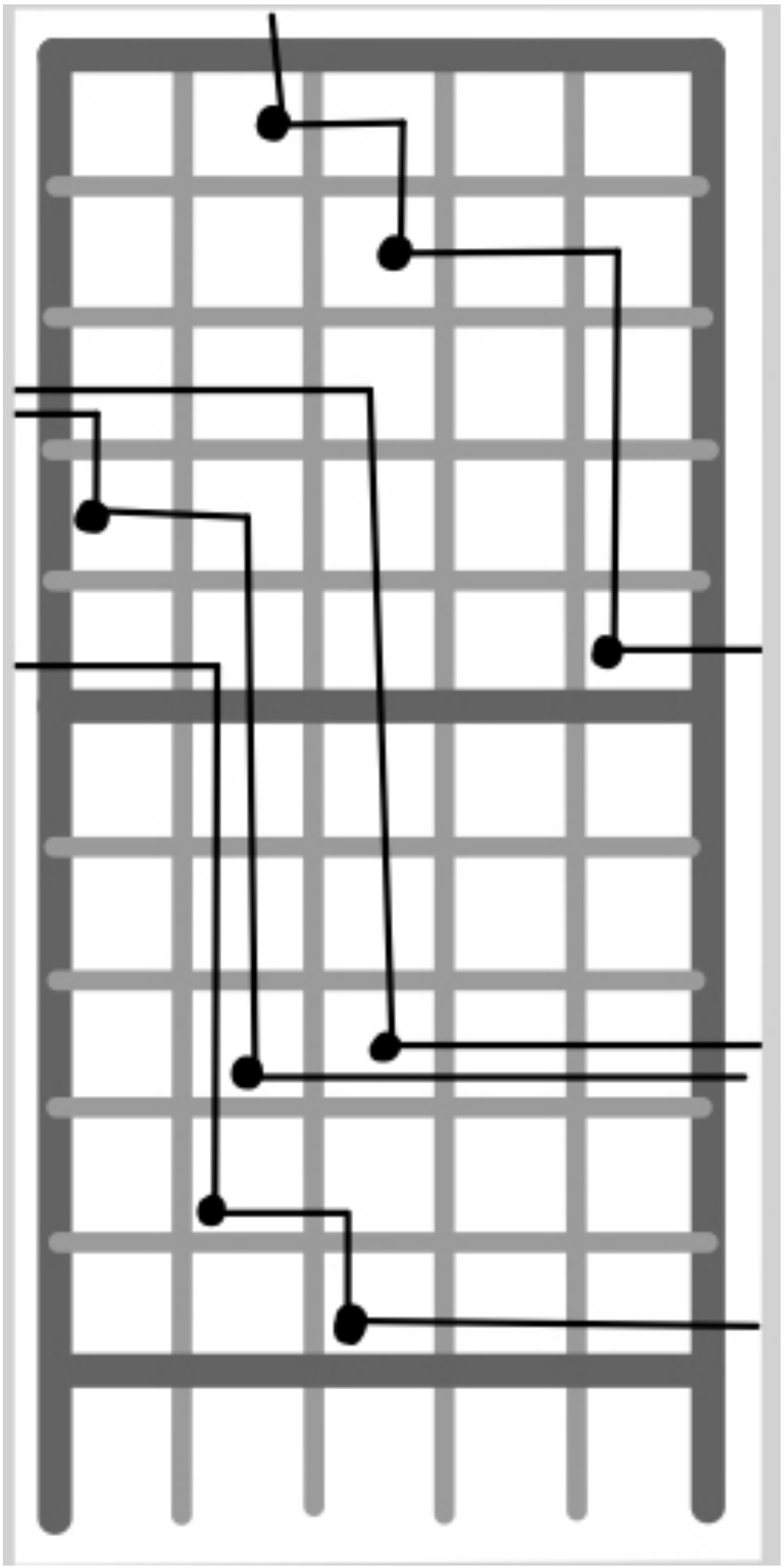
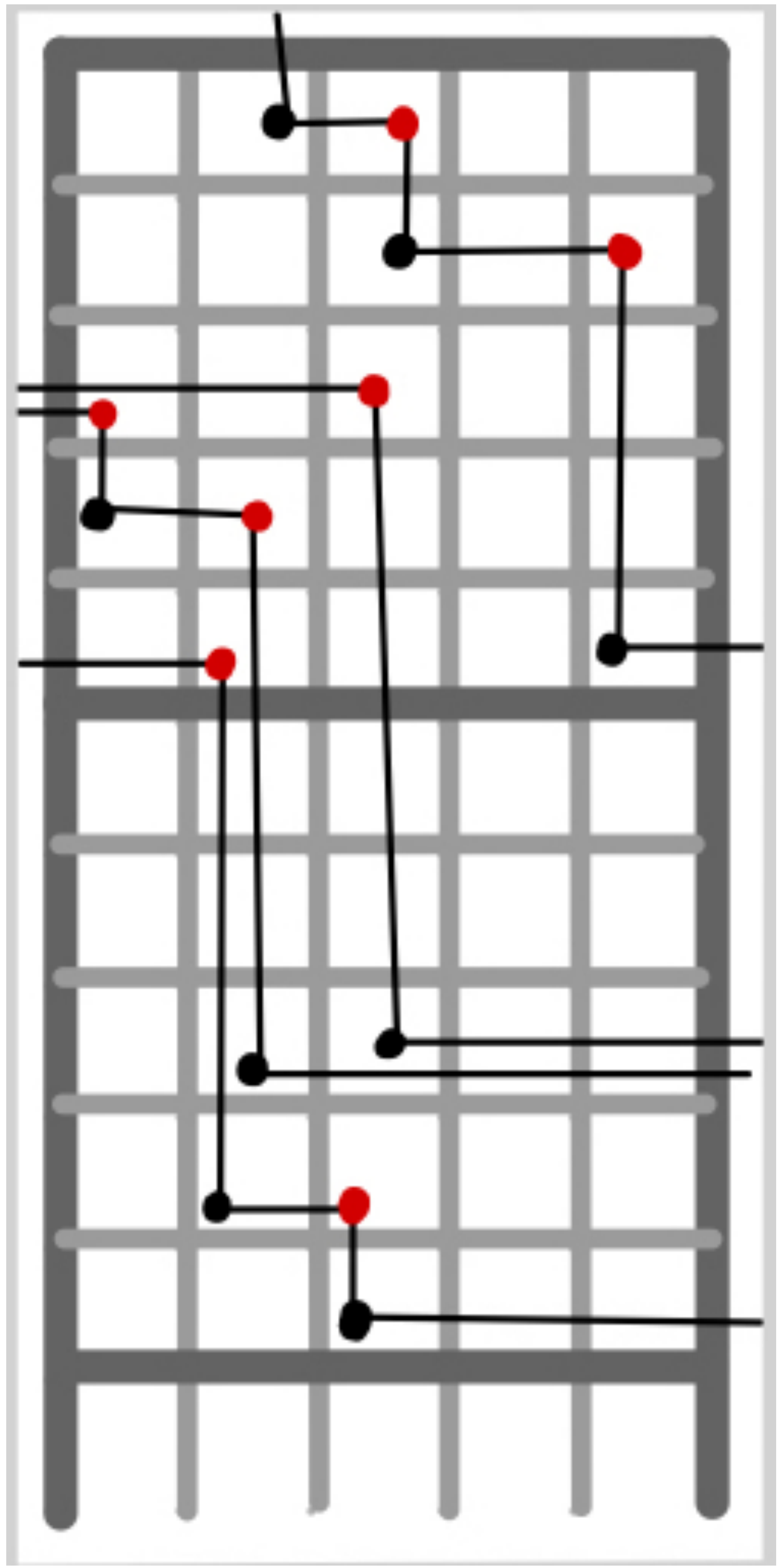


Periodic boundary conditions

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\ldots) & (0,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,1,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \end{pmatrix} \longleftrightarrow \;?$$



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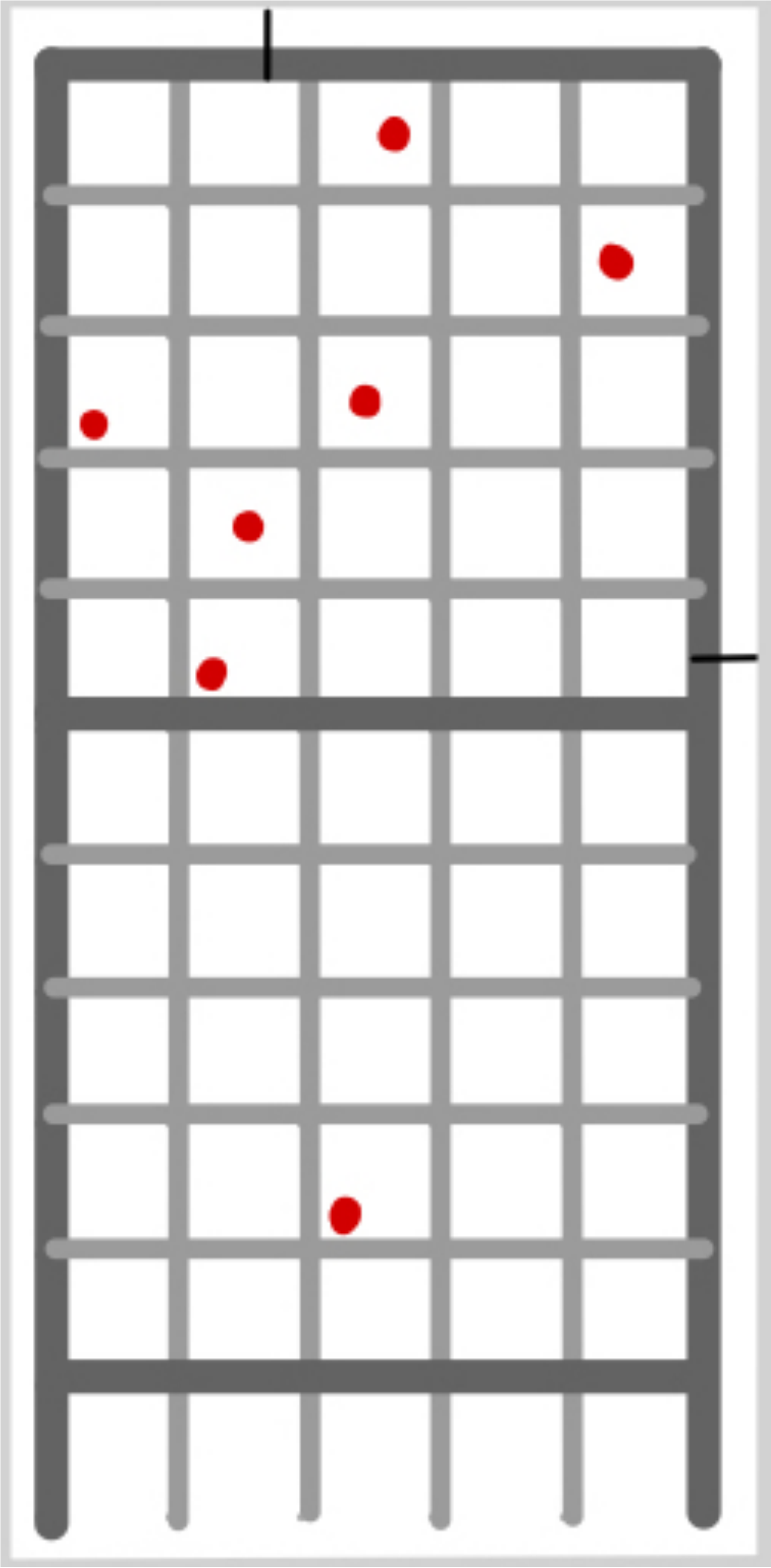
$$\begin{pmatrix} (0,0,\ldots) & (0,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,1,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \end{pmatrix} \longleftrightarrow \; ?$$

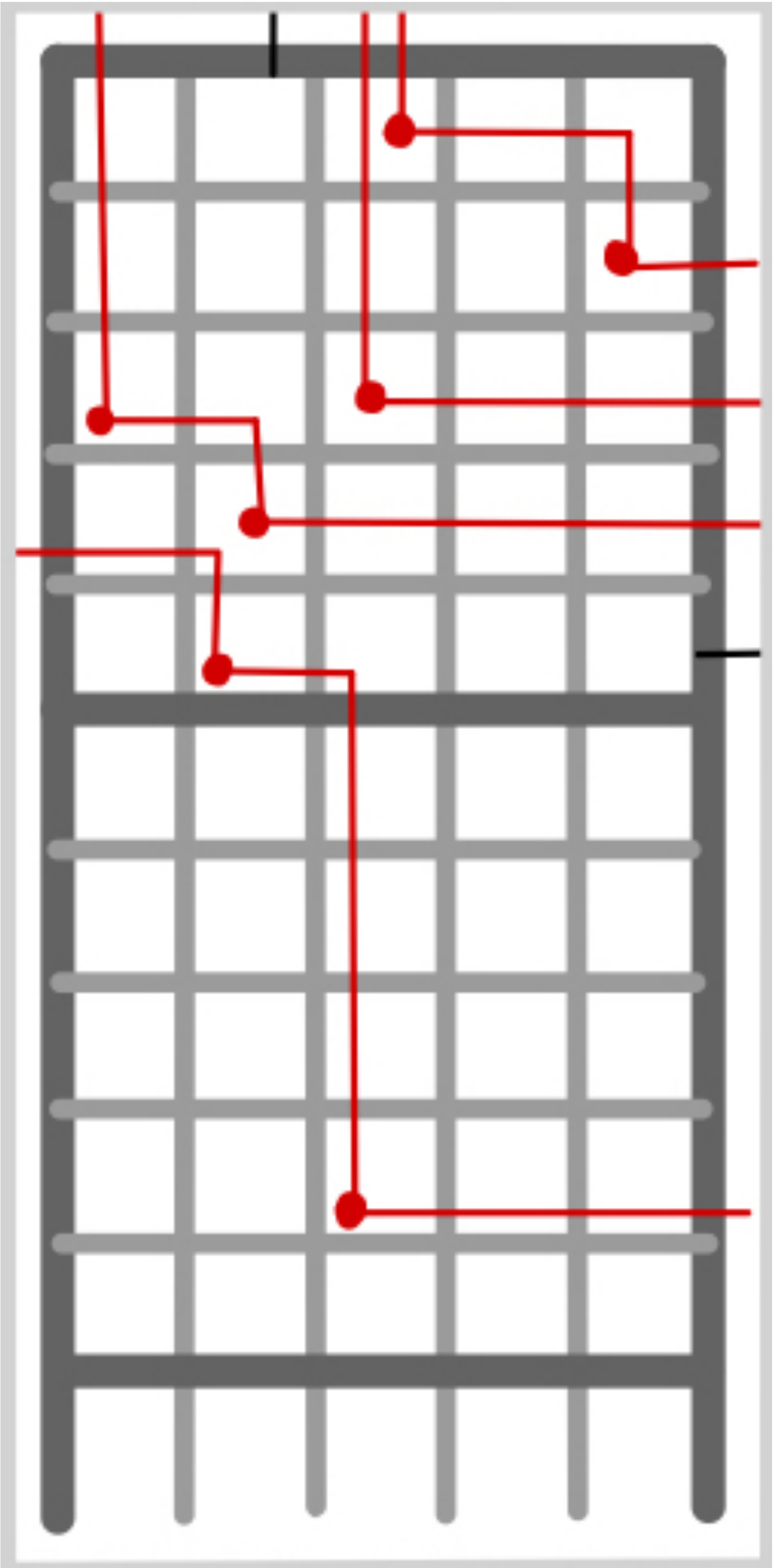


Periodic boundary conditions

Construction of Υ

$$(M^k_{i,j}) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\ldots) & (0,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,1,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \end{pmatrix} \longleftrightarrow \quad ?$$



Periodic boundary conditions

Construction of Υ

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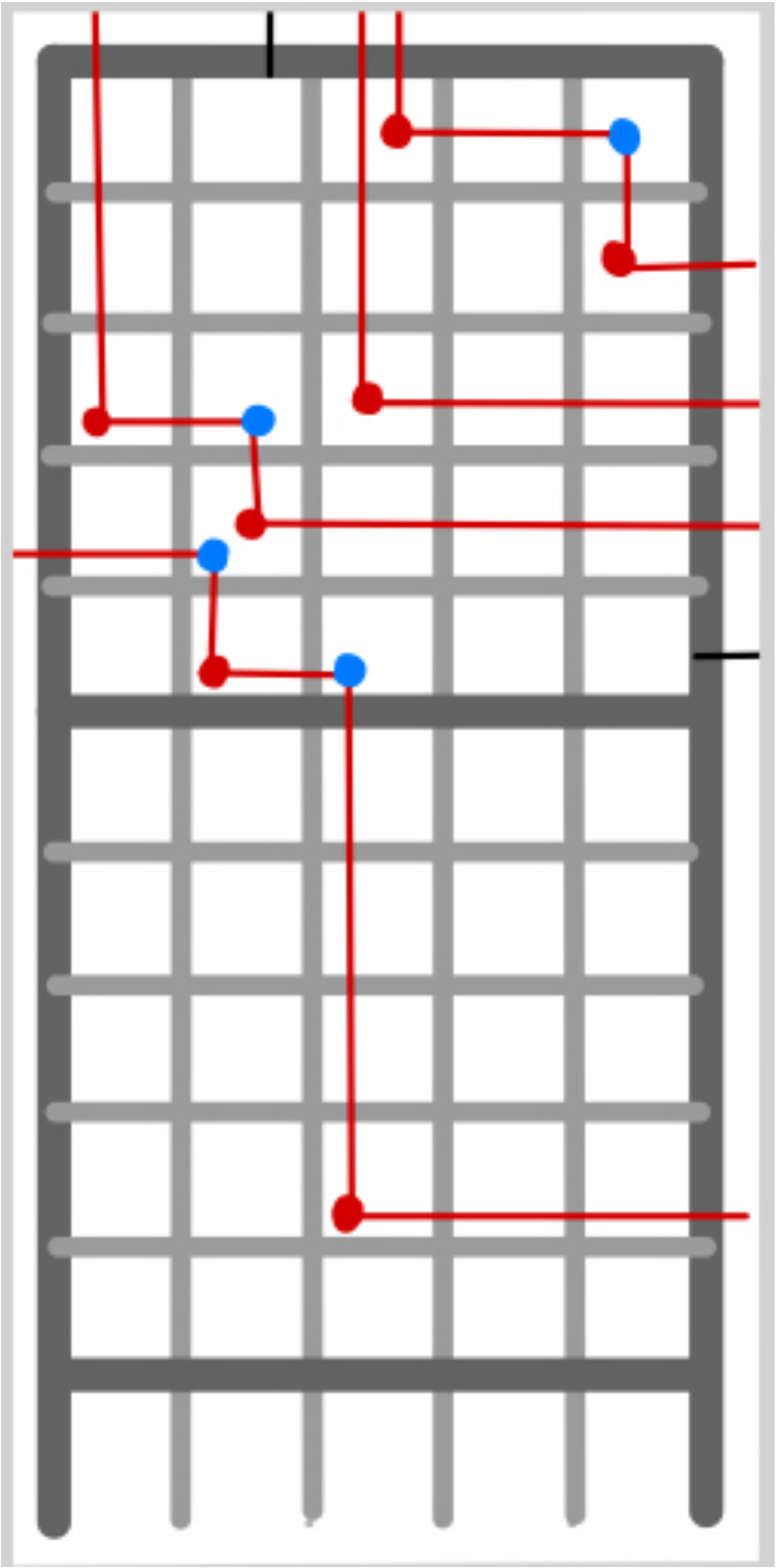
$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix} \longleftrightarrow \quad ?$$



Periodic boundary conditions

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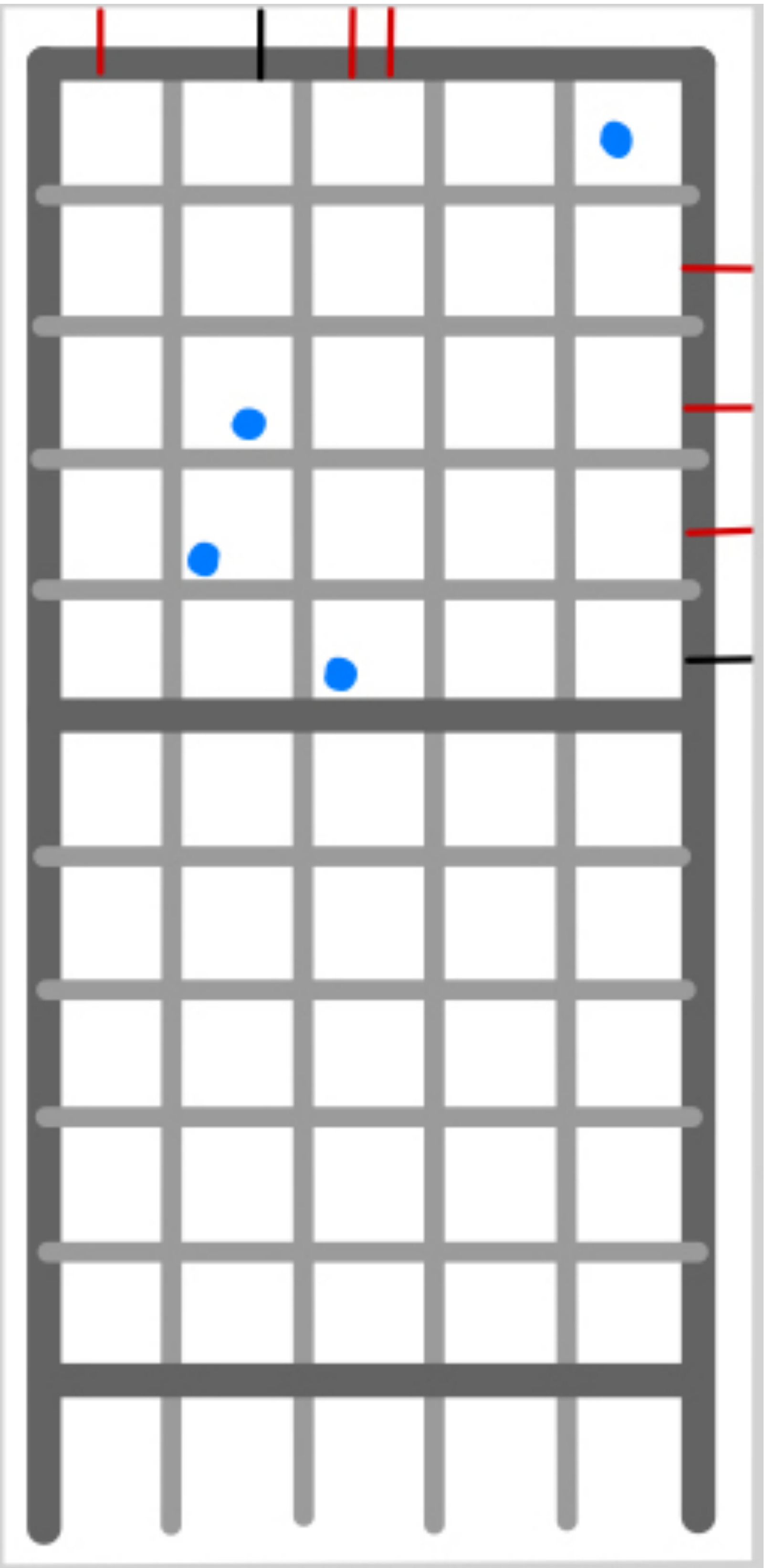

Periodic boundary conditions

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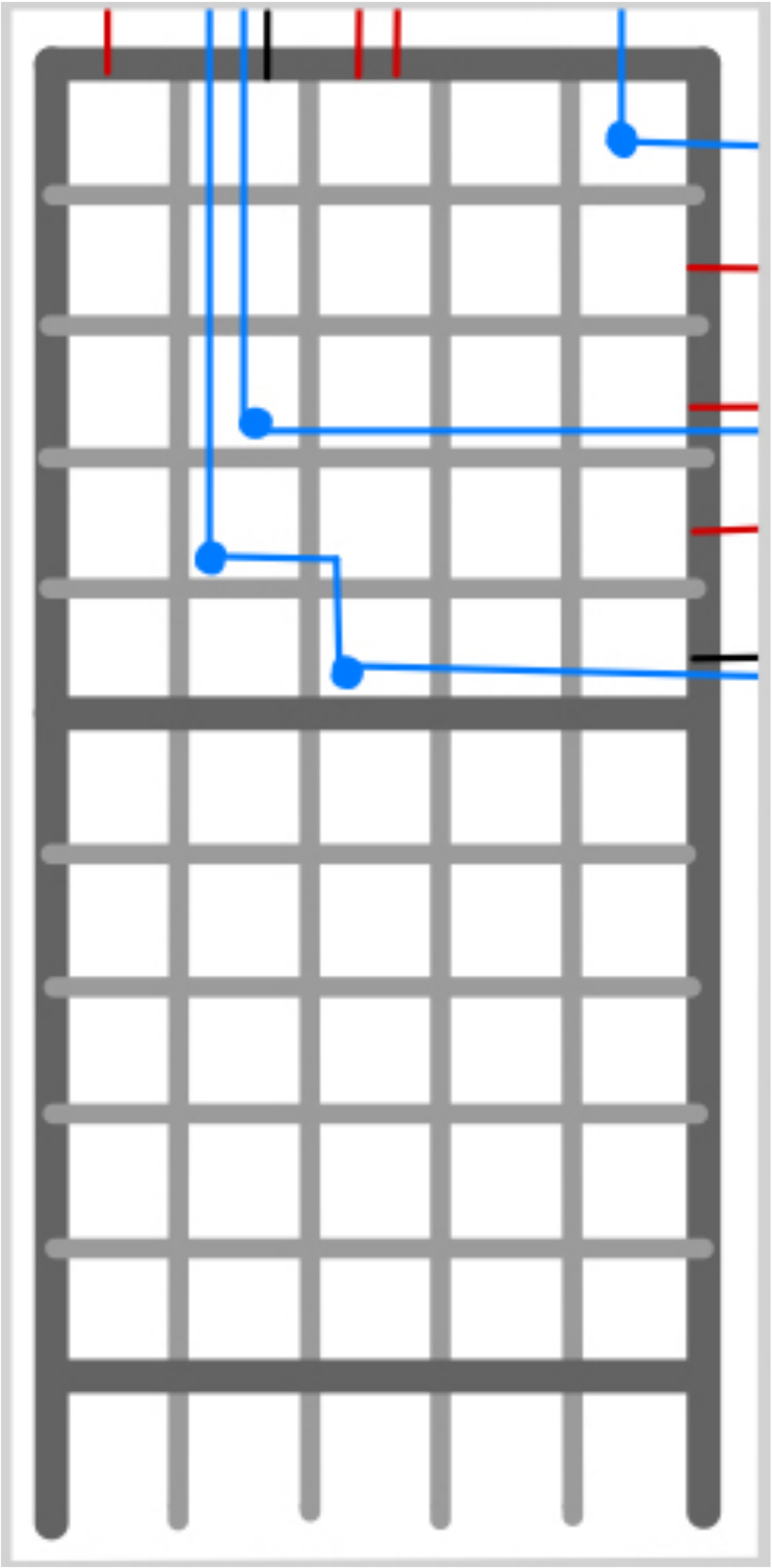
$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix} \longleftrightarrow \quad ?$$

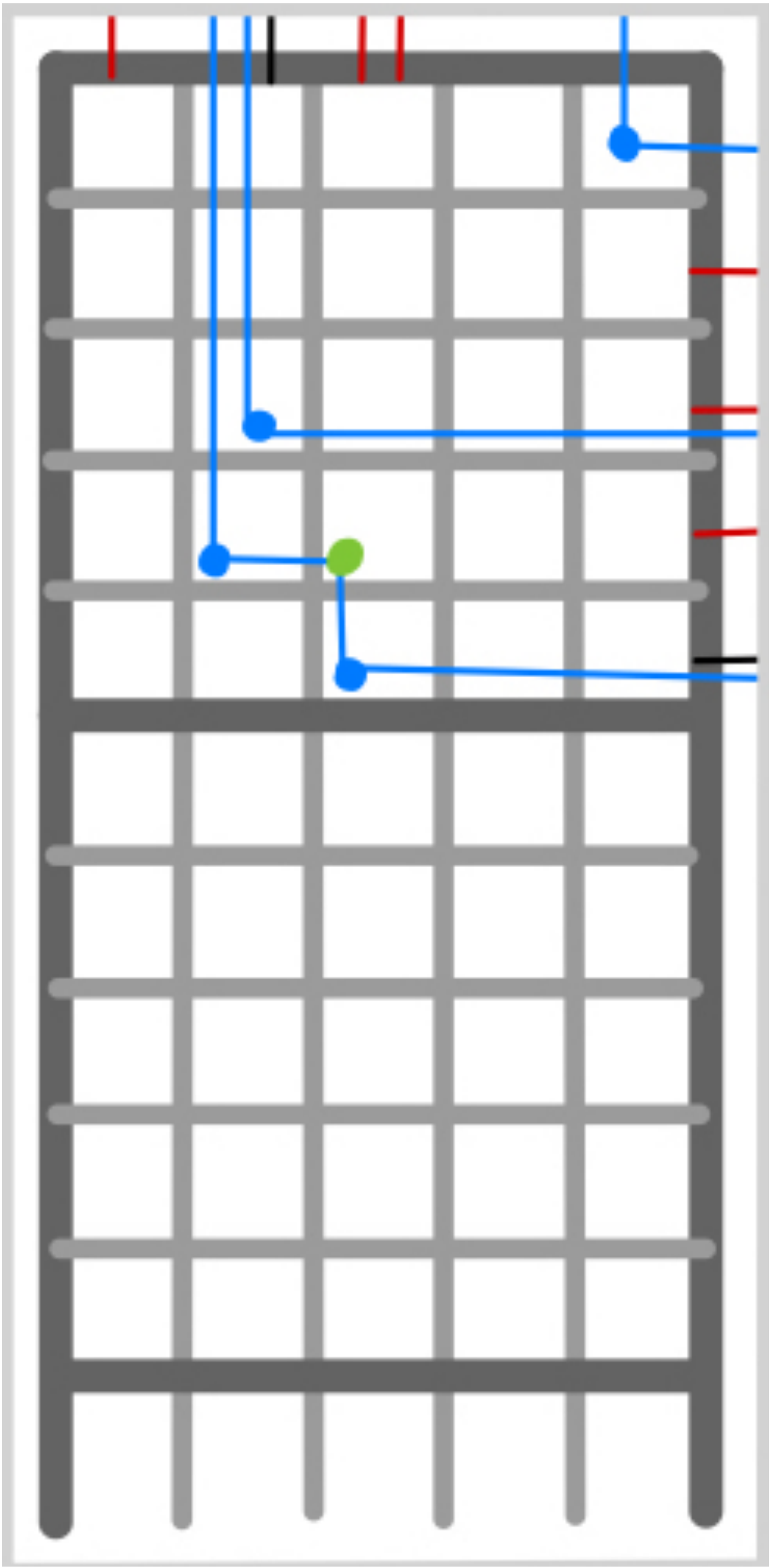


Periodic boundary conditions

Construction of Υ

$$(M^k_{i,j}) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\ldots) & (0,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,1,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \end{pmatrix} \longleftrightarrow \quad ?$$

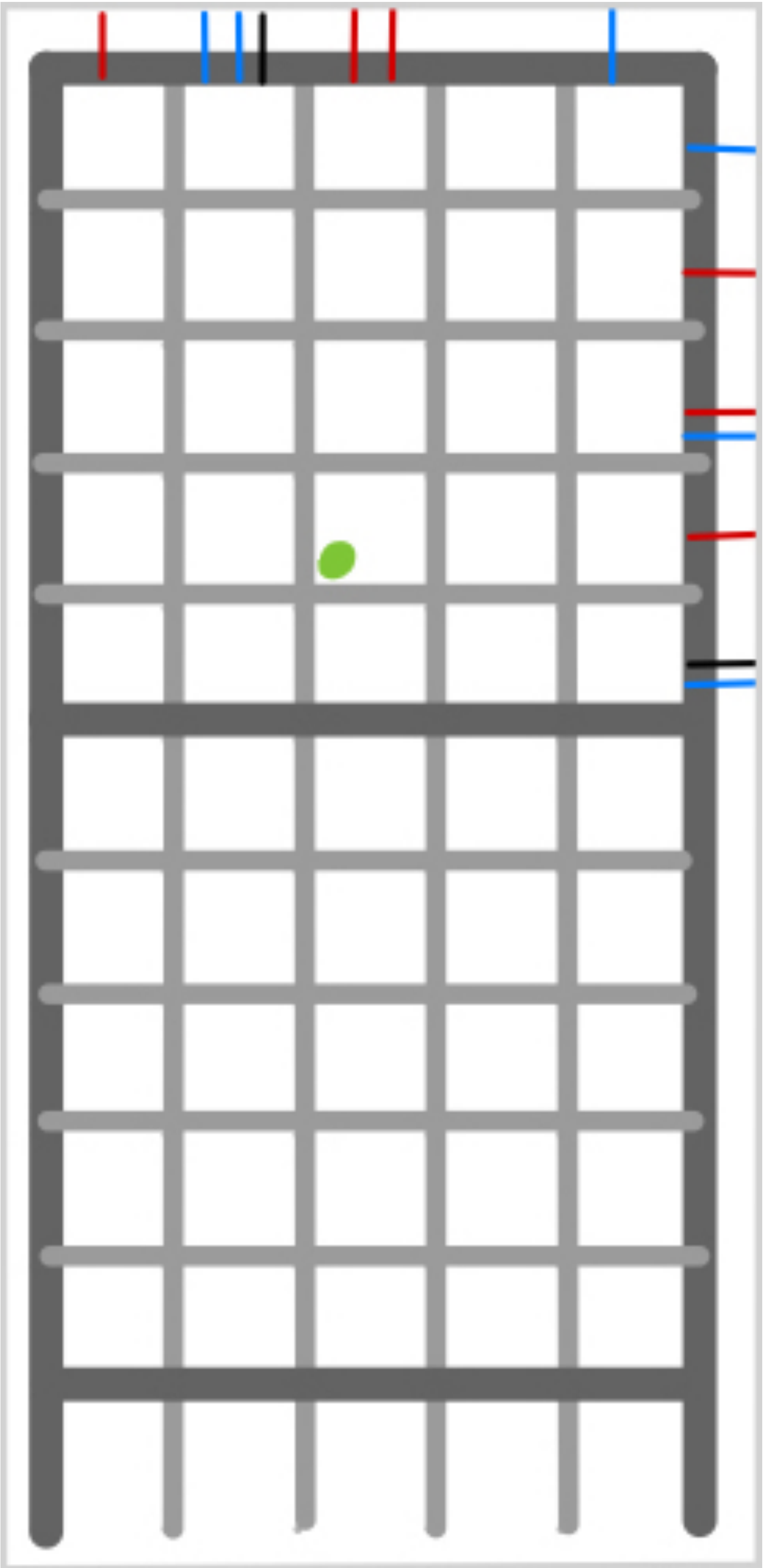


Periodic boundary conditions

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix} \longleftrightarrow \quad ?$$

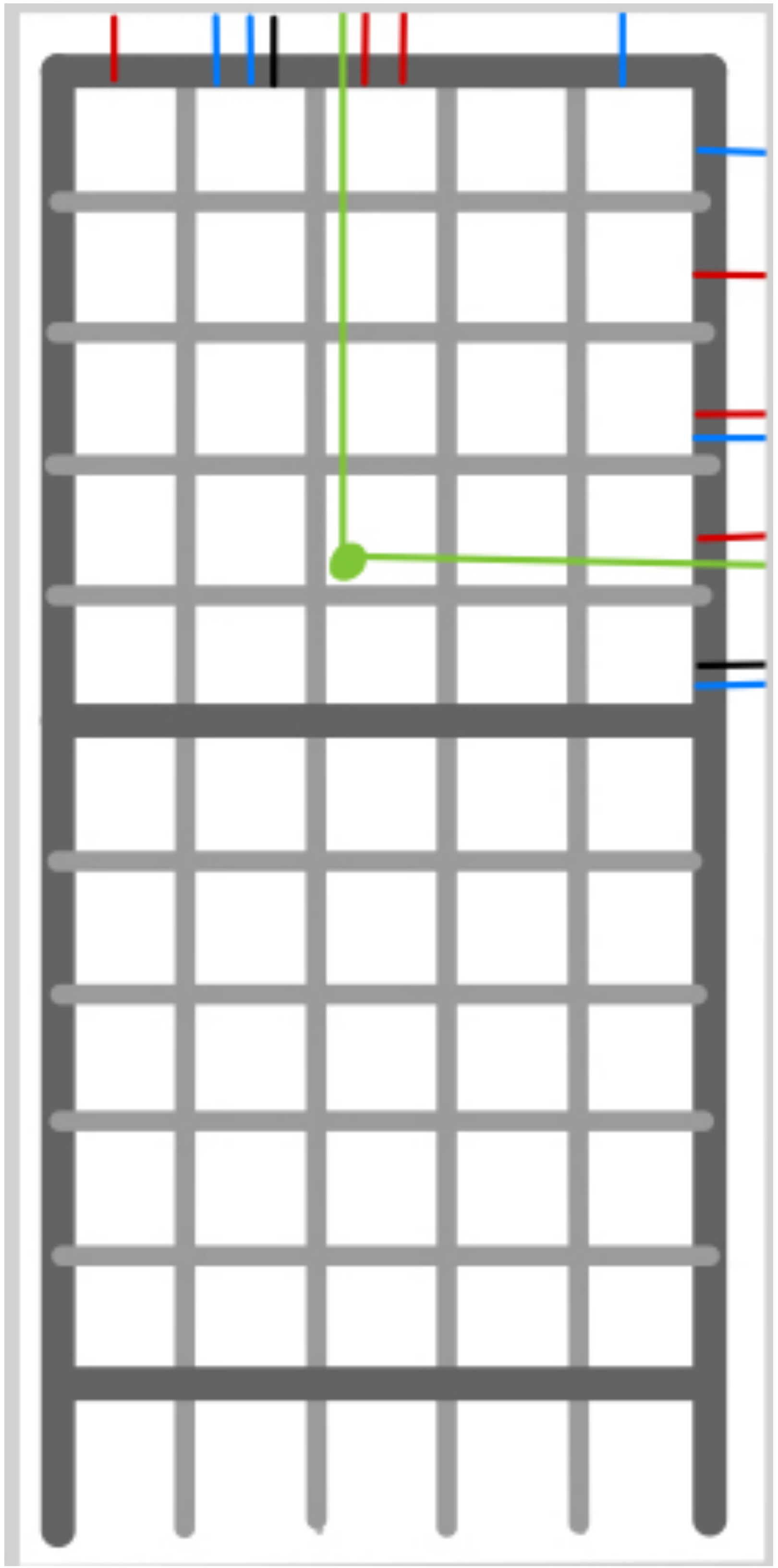


Periodic boundary conditions

Construction of Υ

$$(M^k_{i,j}) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix} \longleftrightarrow \quad ?$$

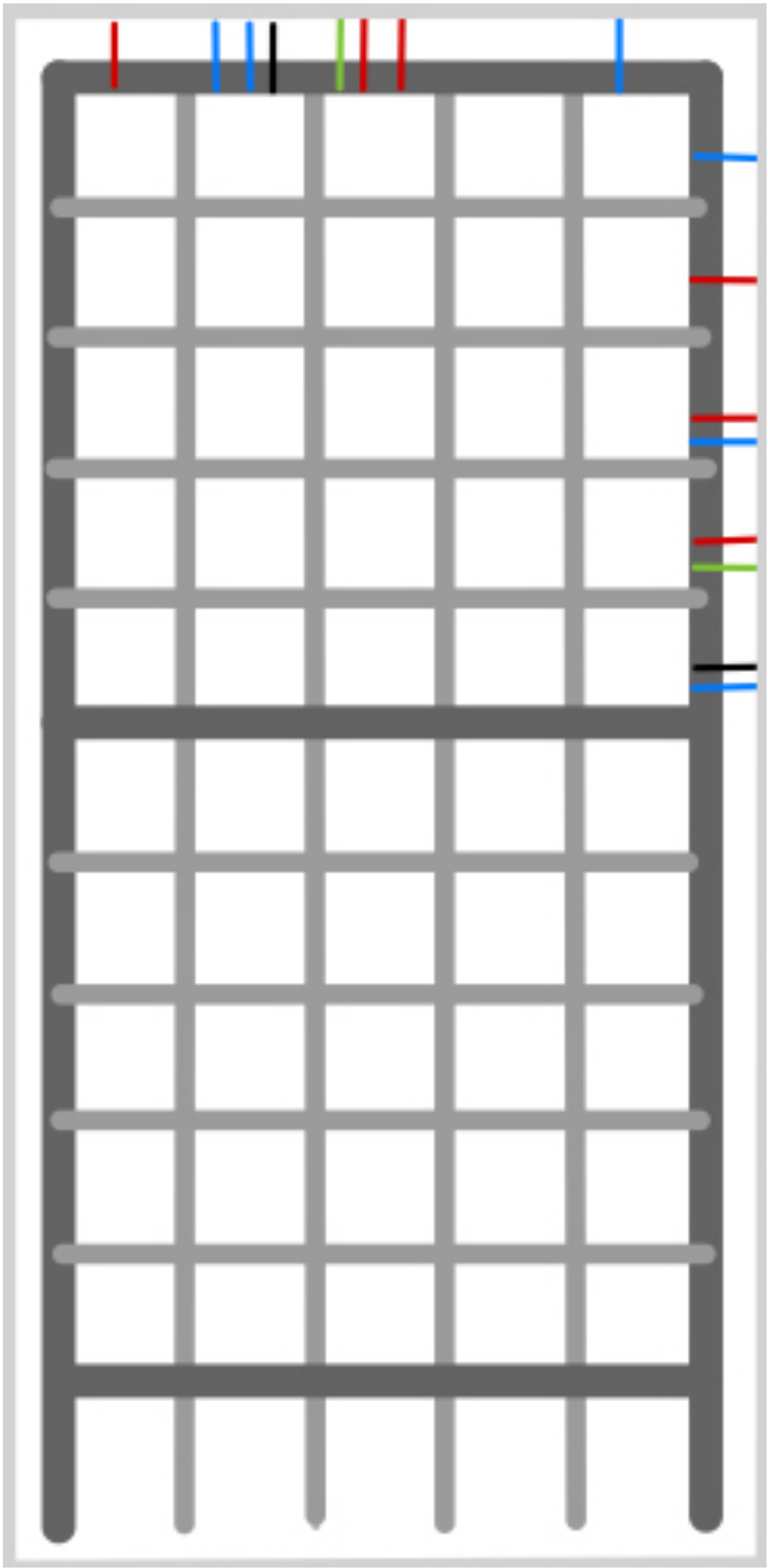


Periodic boundary conditions

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix} \longleftrightarrow \; ?$$

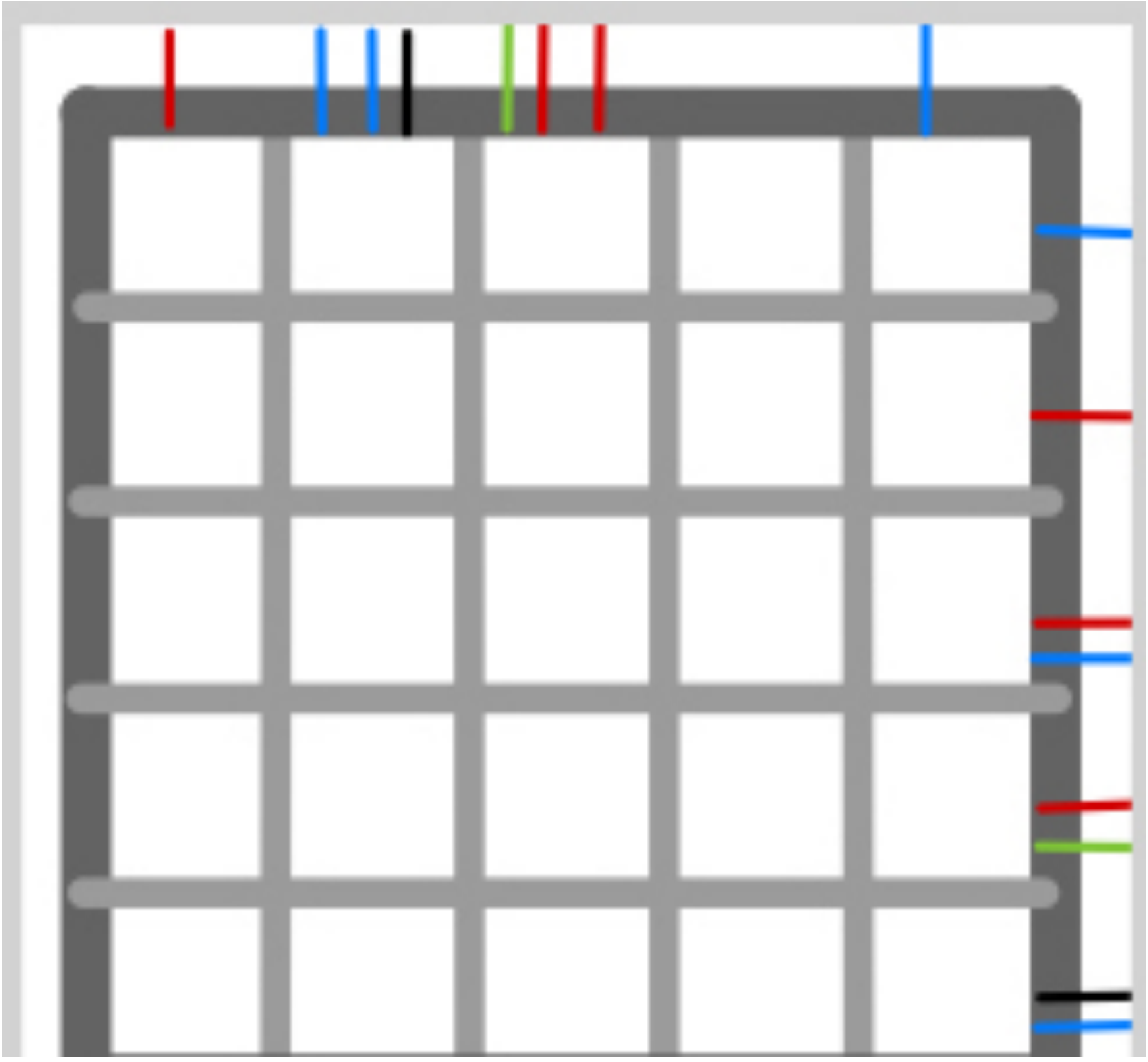


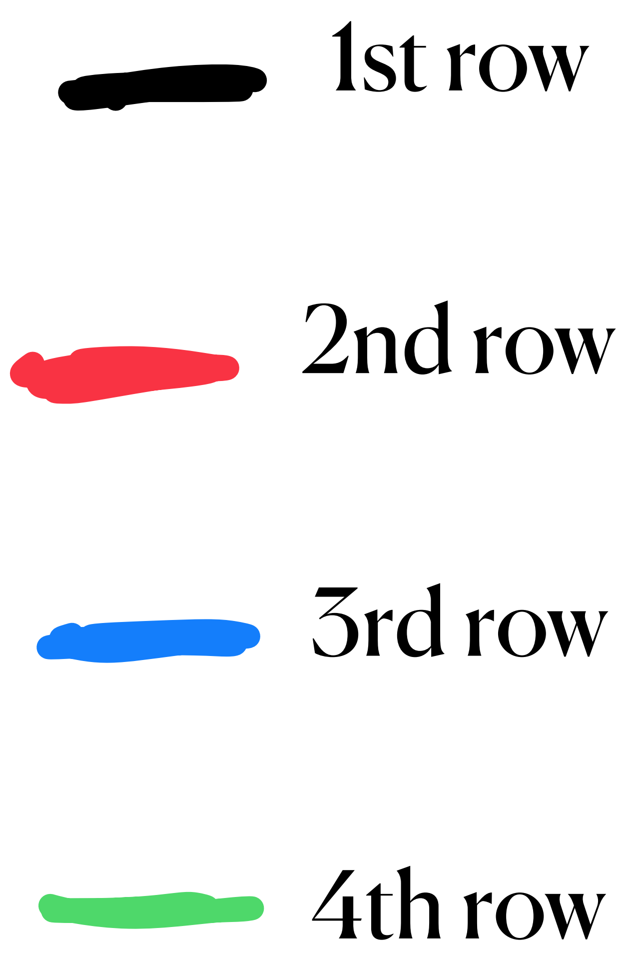
Periodic boundary conditions

Construction of Υ

$$(M^k_{i,j}) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\ldots) & (0,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,1,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \end{pmatrix} \longleftrightarrow \quad ?$$

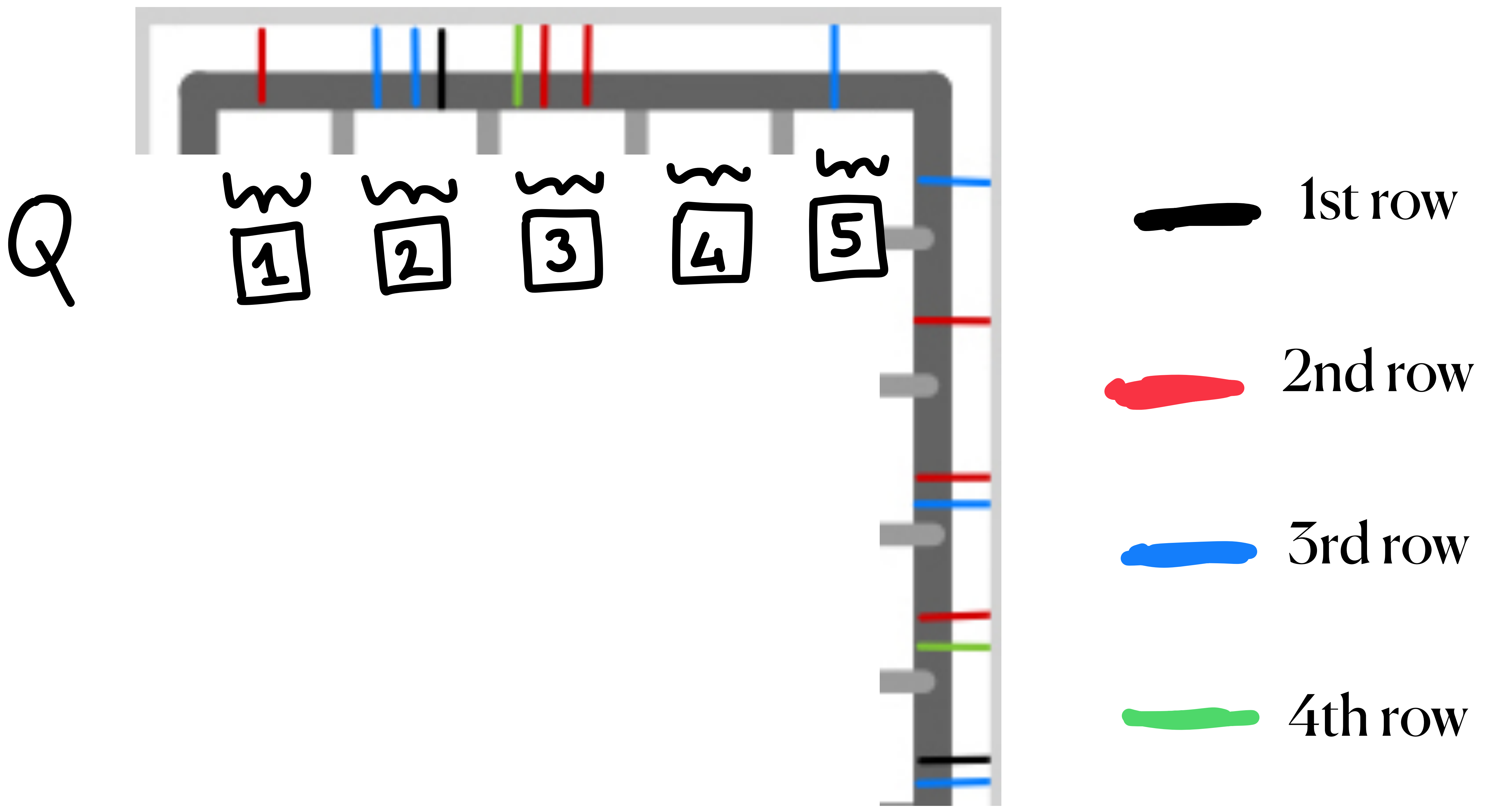




Construction of Υ

$$(M^k_{i,j}) \longleftrightarrow (V, W; \kappa)$$

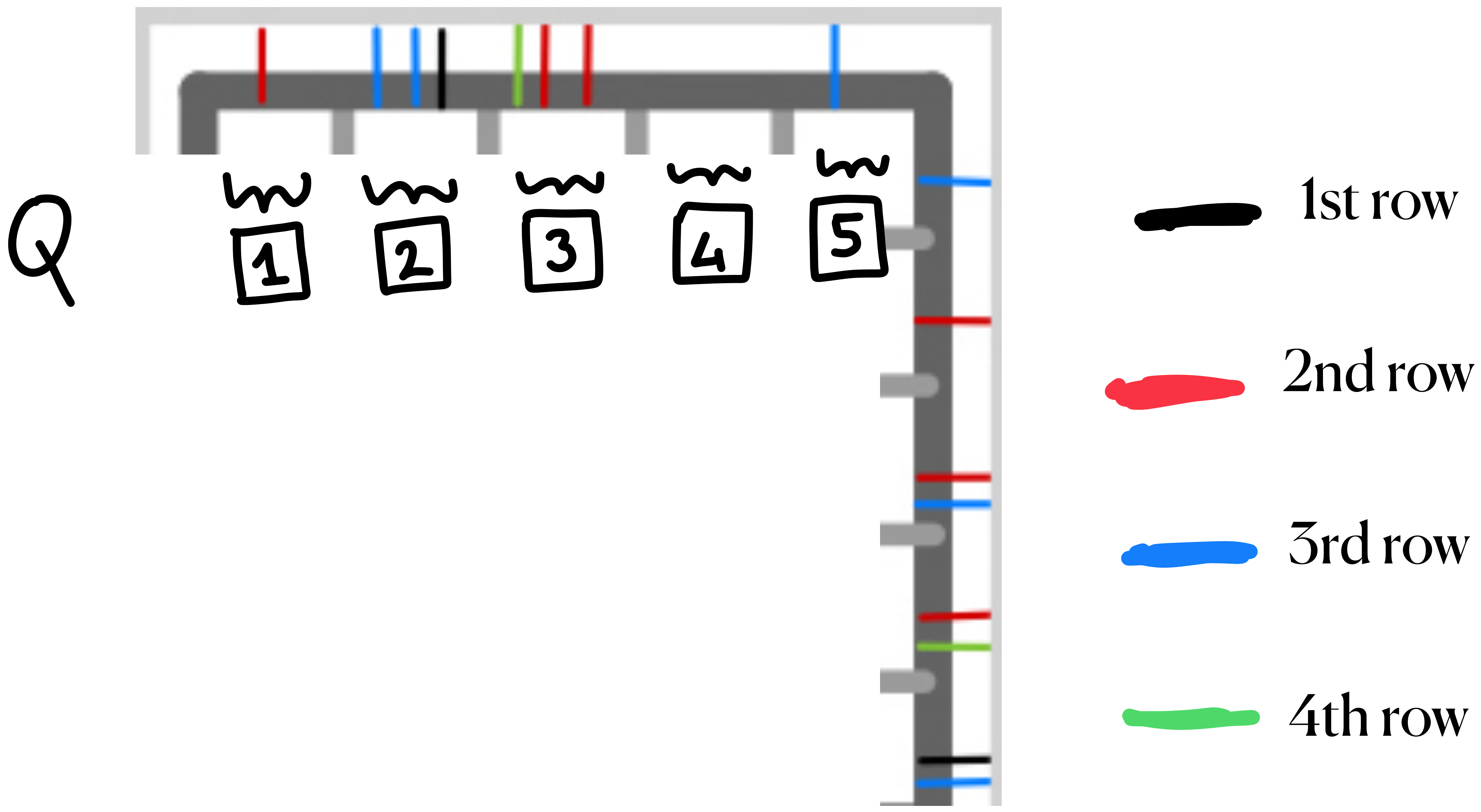
$$\begin{pmatrix} (0,0,\ldots) & (0,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,1,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \end{pmatrix} \longleftrightarrow \ ?$$

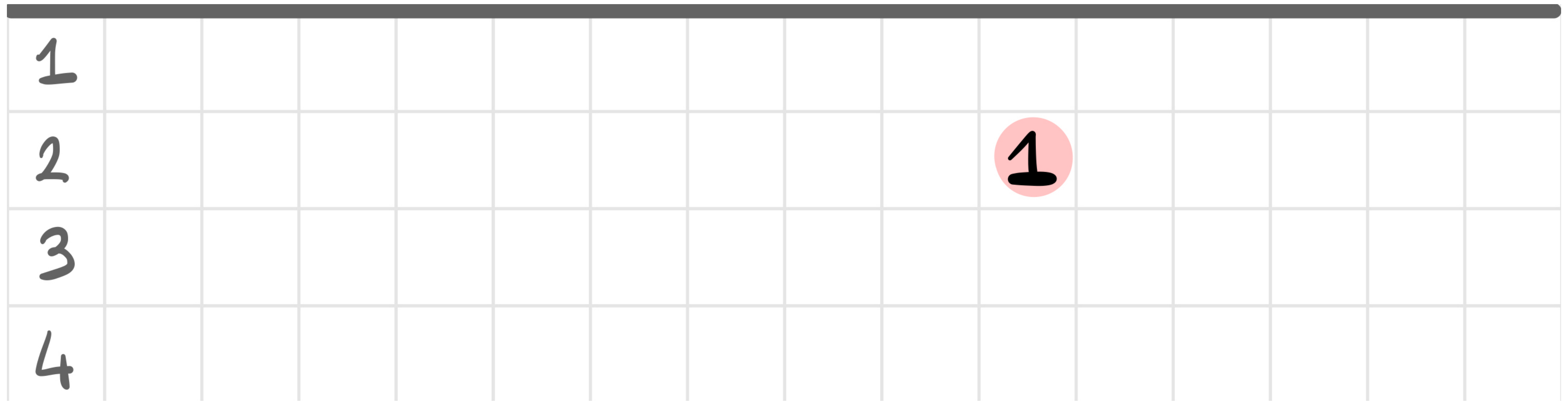


Construction of Υ

$$(M^k_{i,j}) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\ldots) & (0,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,1,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \end{pmatrix} \longleftrightarrow \; ?$$

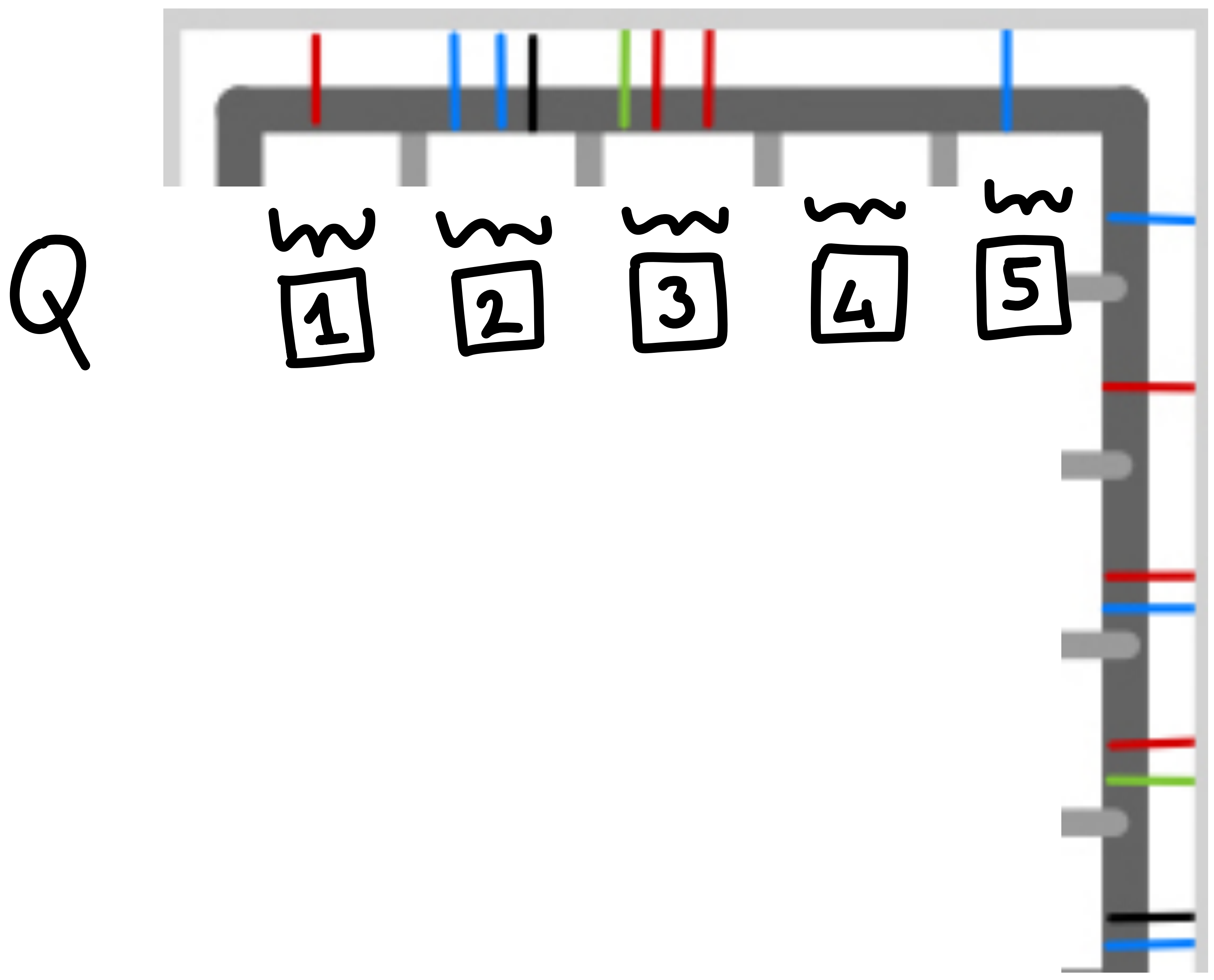




Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\ldots) & (0,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,1,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \end{pmatrix} \longleftrightarrow ?$$

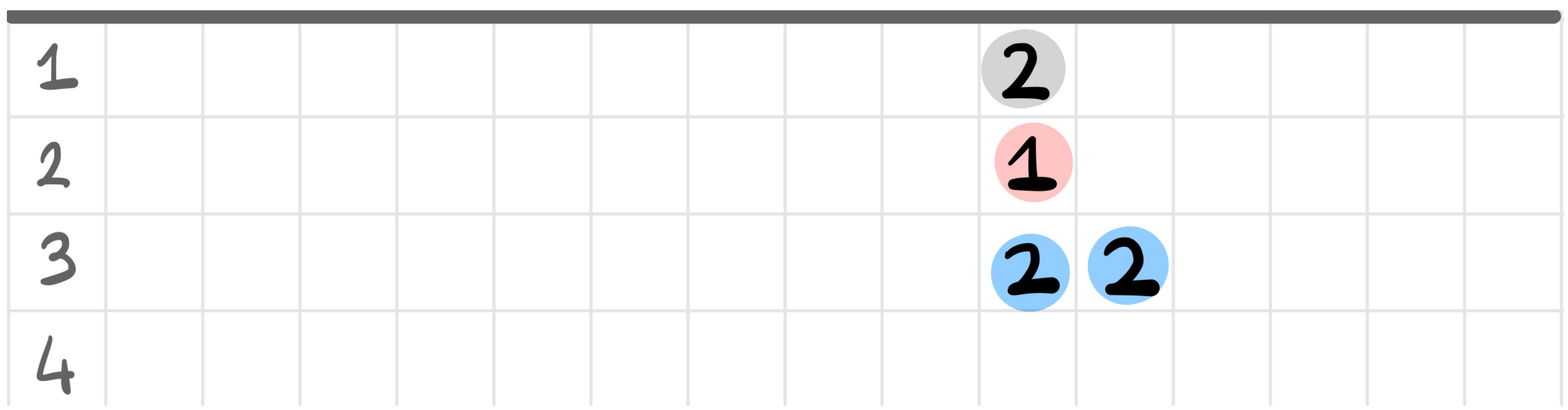


1st row

2nd row

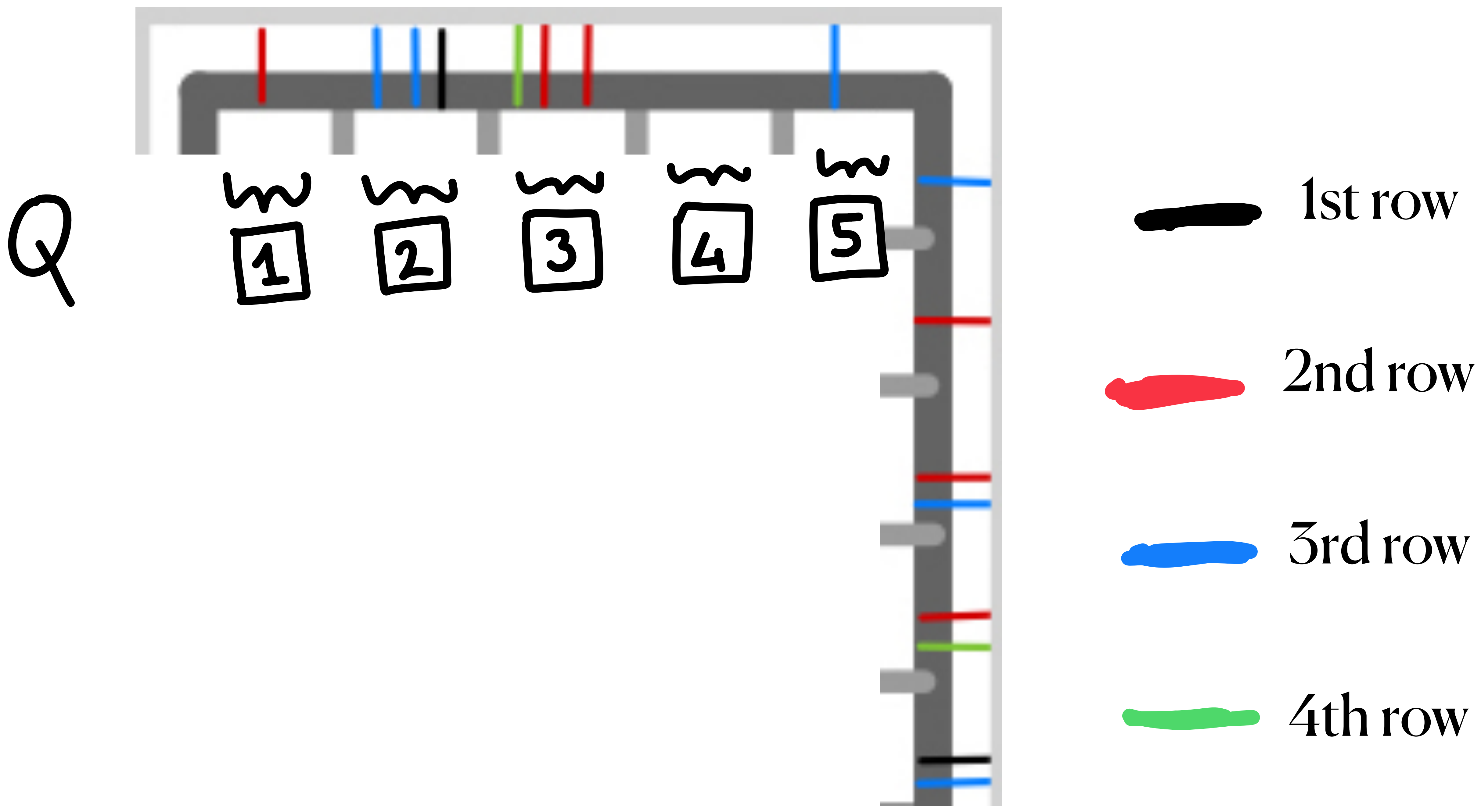
3rd row

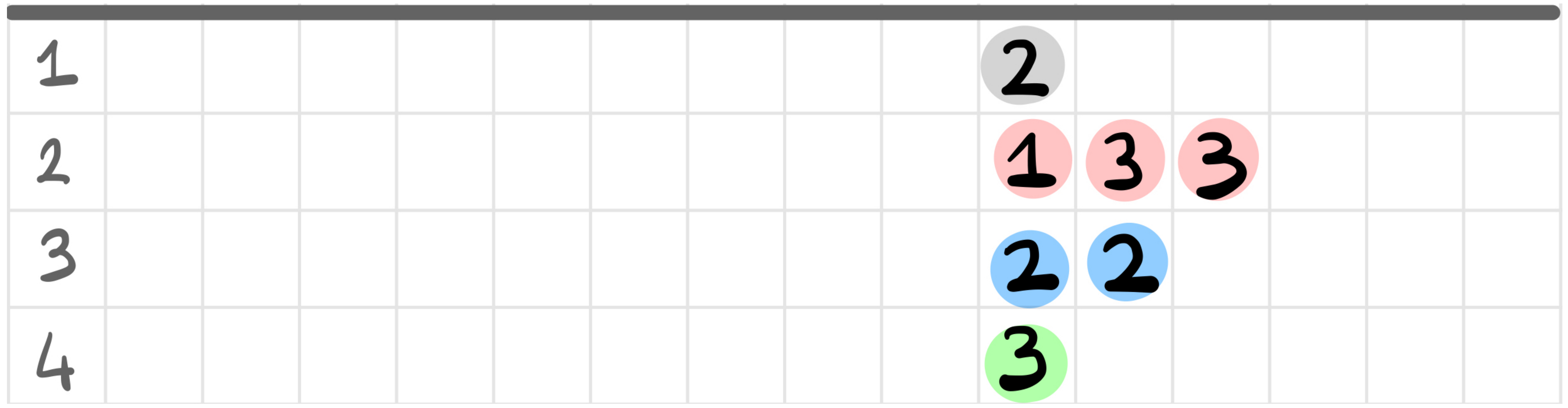
4th row



Construction of Υ

$$(M^k_{i,j}) \longleftrightarrow (V, W; \kappa) \qquad \begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix} \longleftrightarrow \ ?$$

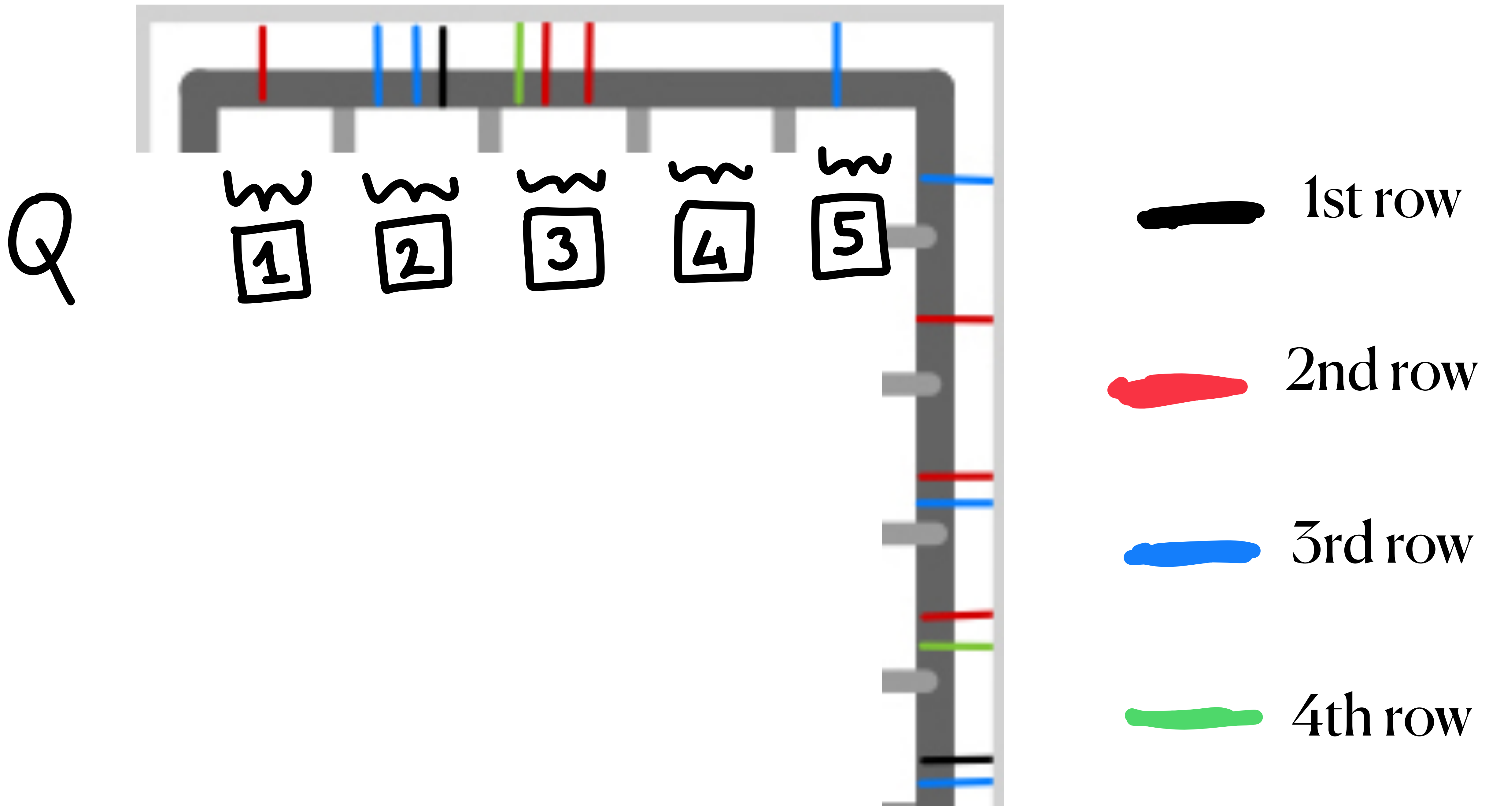


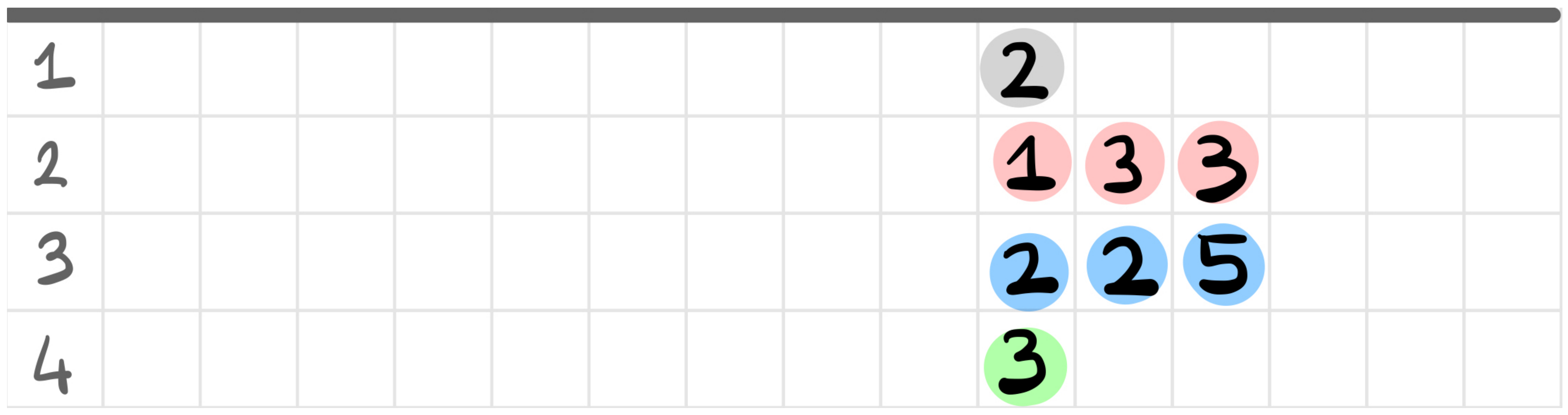


Construction of Υ

$$(M^k_{i,j}) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix} \longleftrightarrow \ ?$$

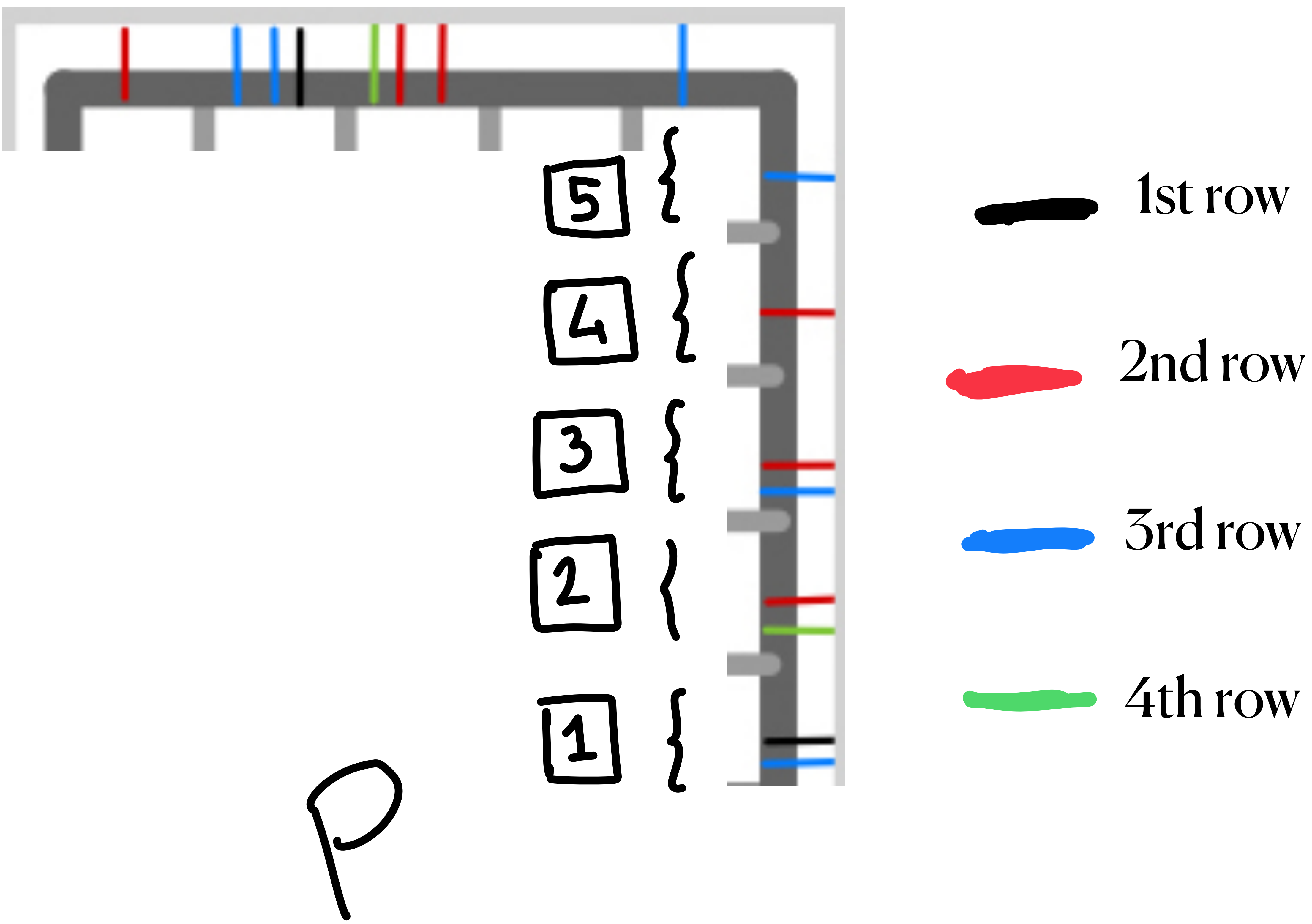


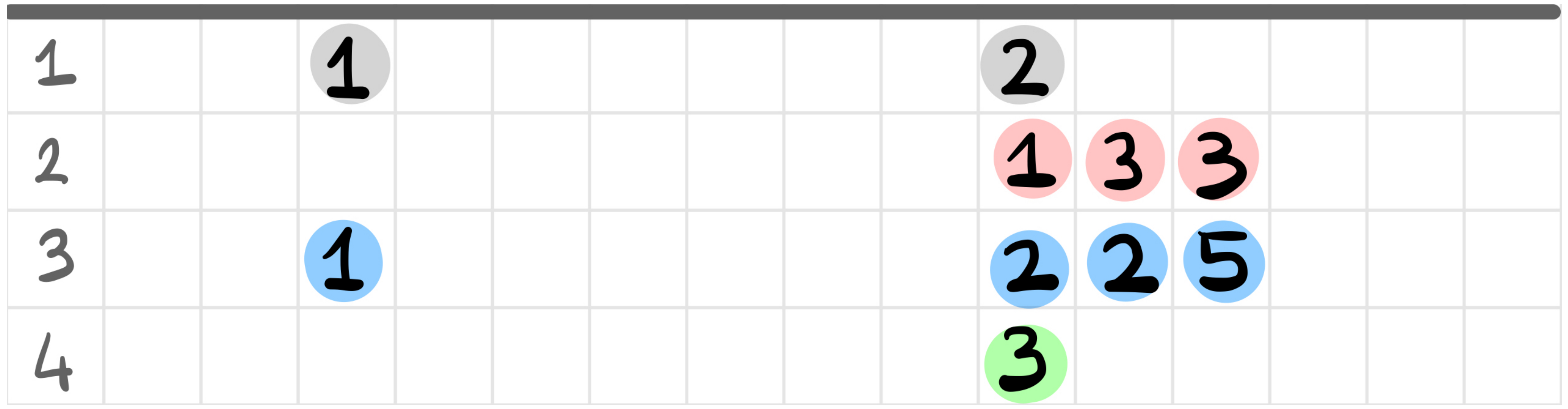


Construction of Υ

$$(M^k_{i,j}) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\ldots) & (0,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,1,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \end{pmatrix} \longleftrightarrow \ ?$$

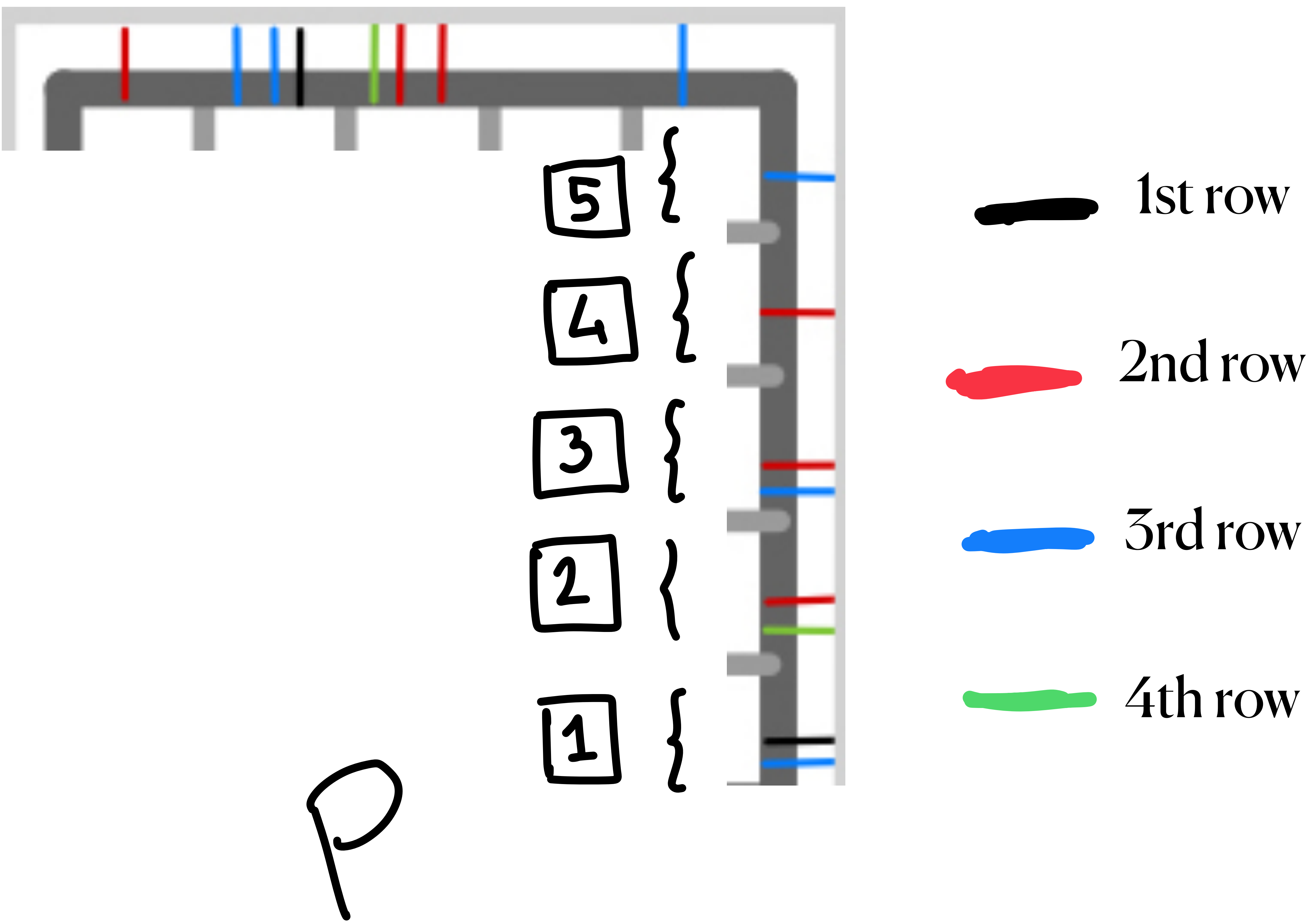


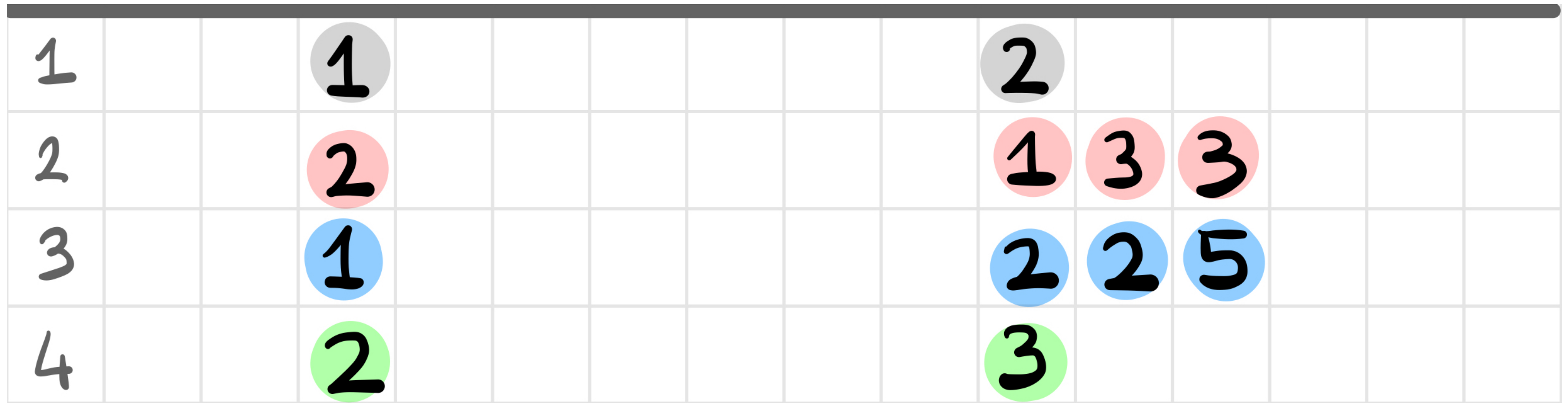


Construction of Υ

$$(M^k_{i,j}) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\ldots) & (0,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,1,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \end{pmatrix} \longleftrightarrow \; ?$$

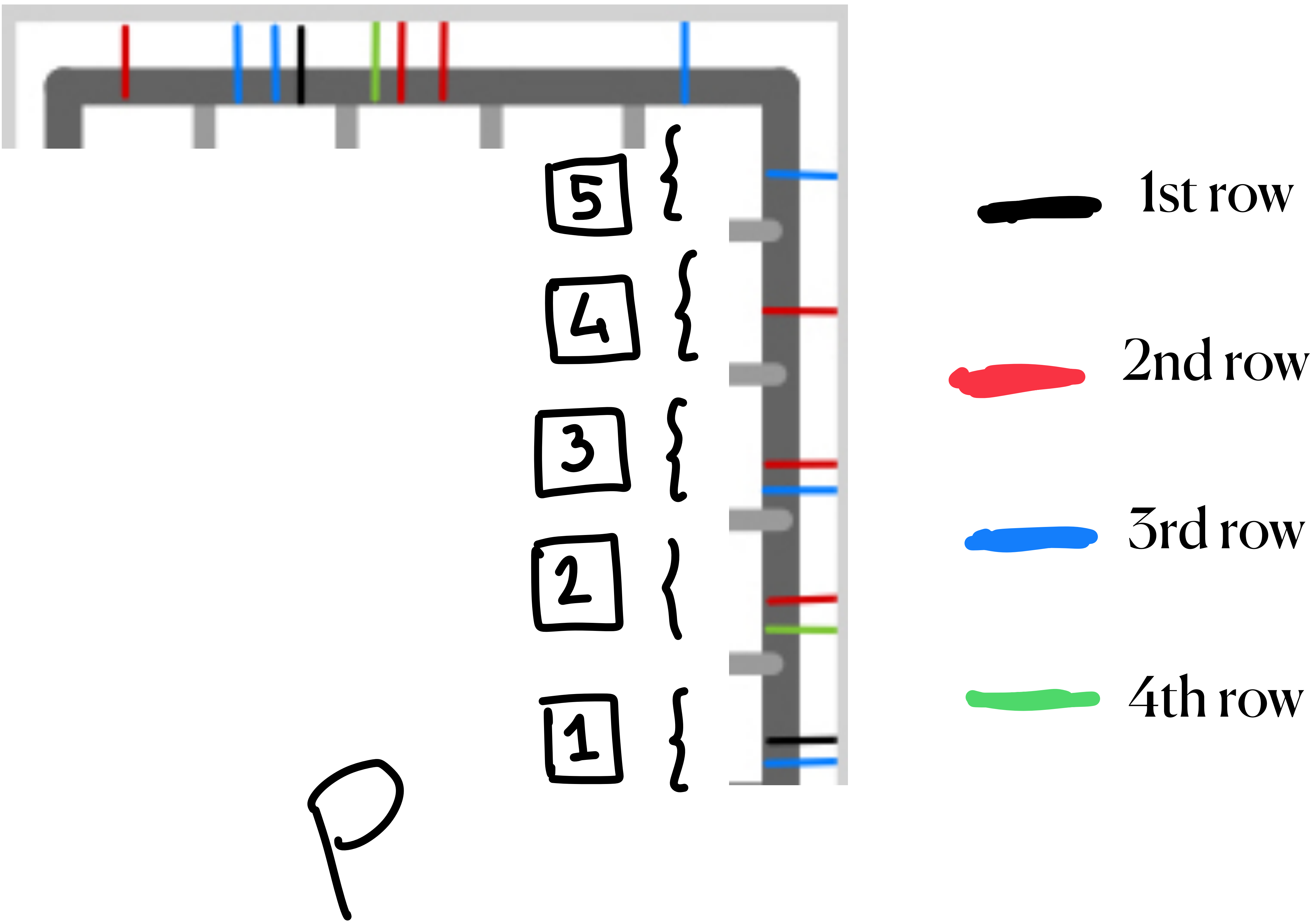


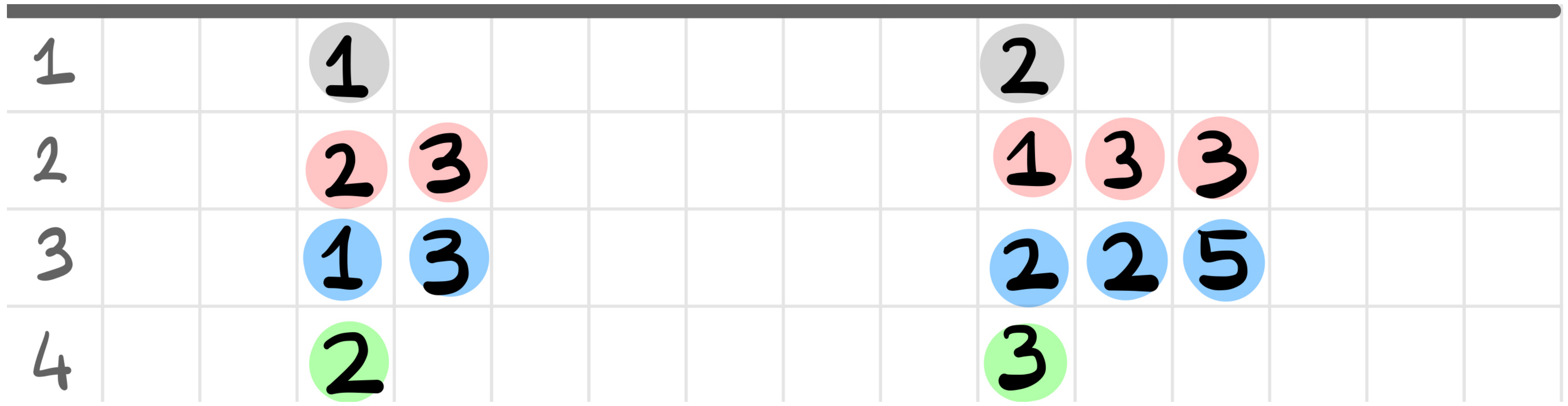


Construction of Υ

$$(M_{i,j}^{k}) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\ldots) & (0,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,1,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \end{pmatrix} \longleftrightarrow \; ?$$

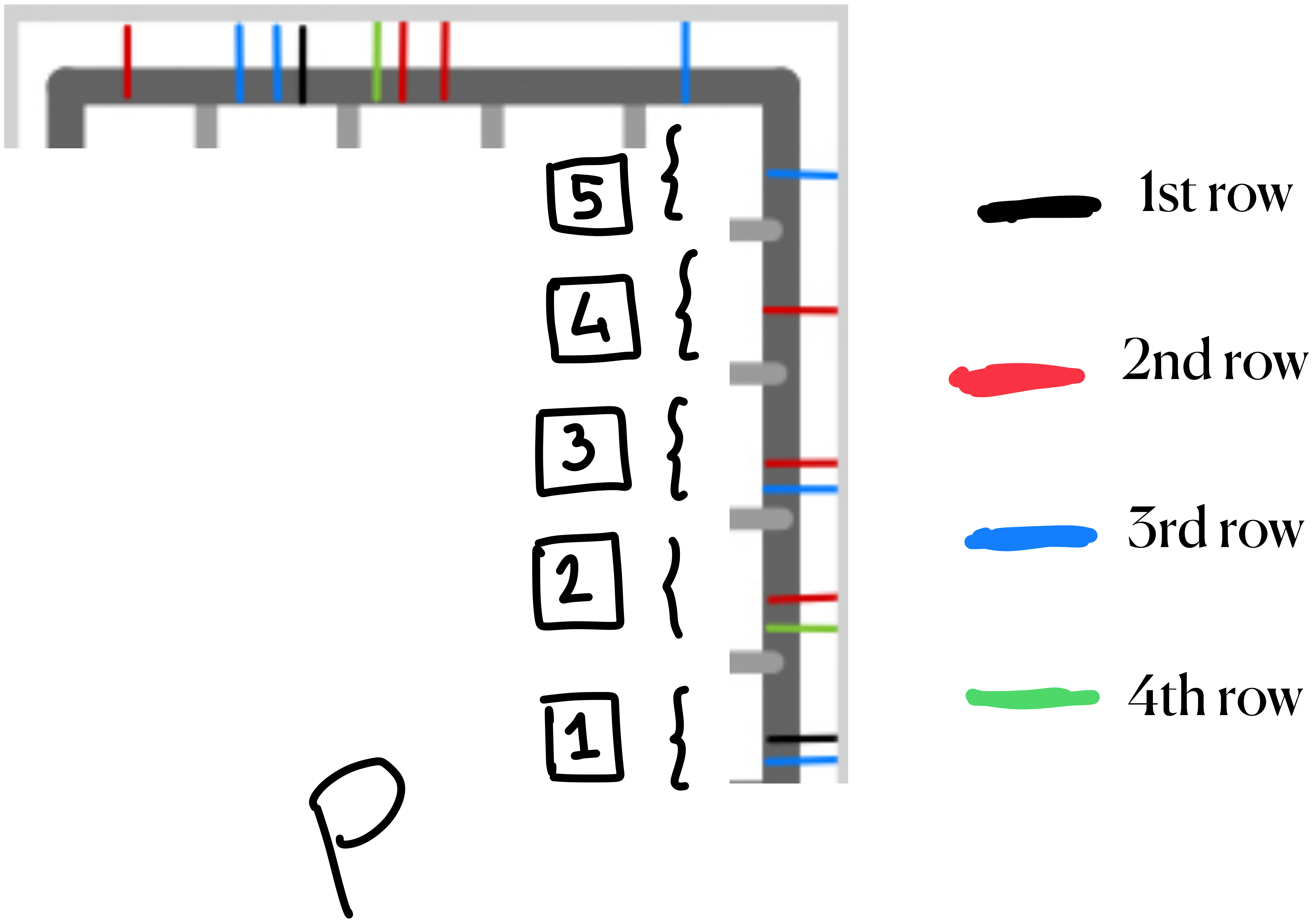


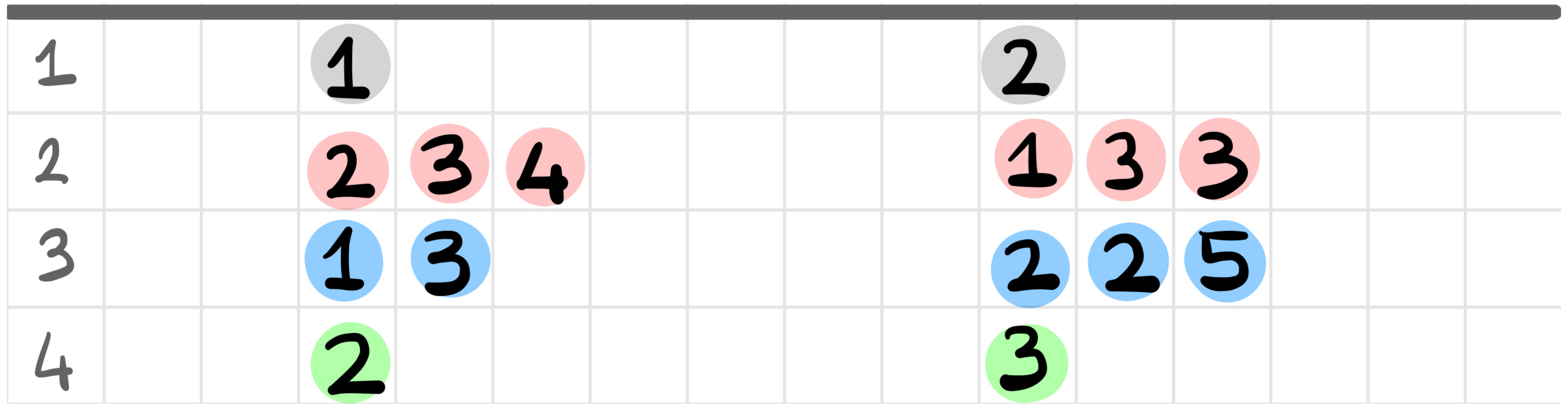


Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\ldots) & (0,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,1,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \end{pmatrix} \longleftrightarrow ?$$

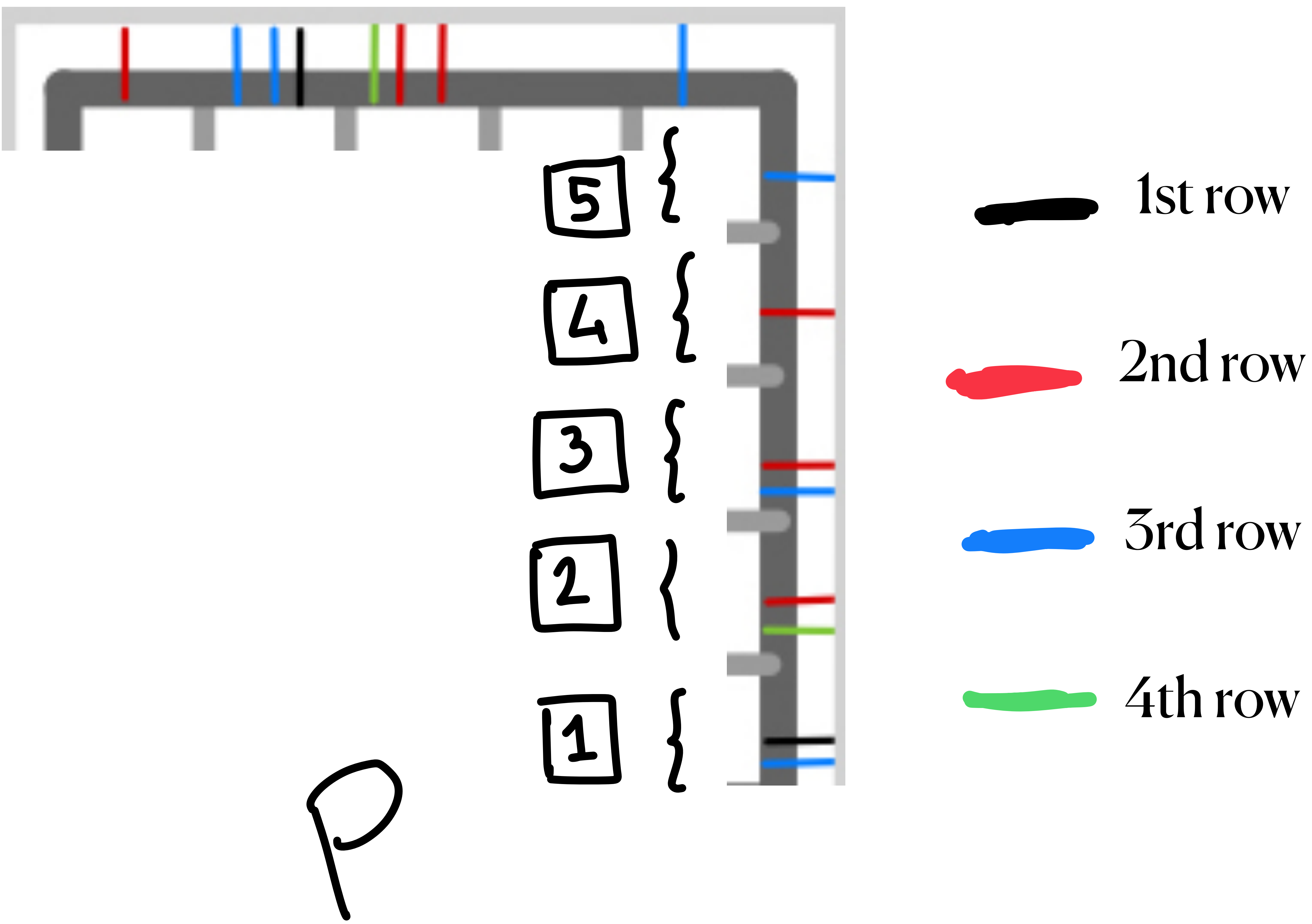


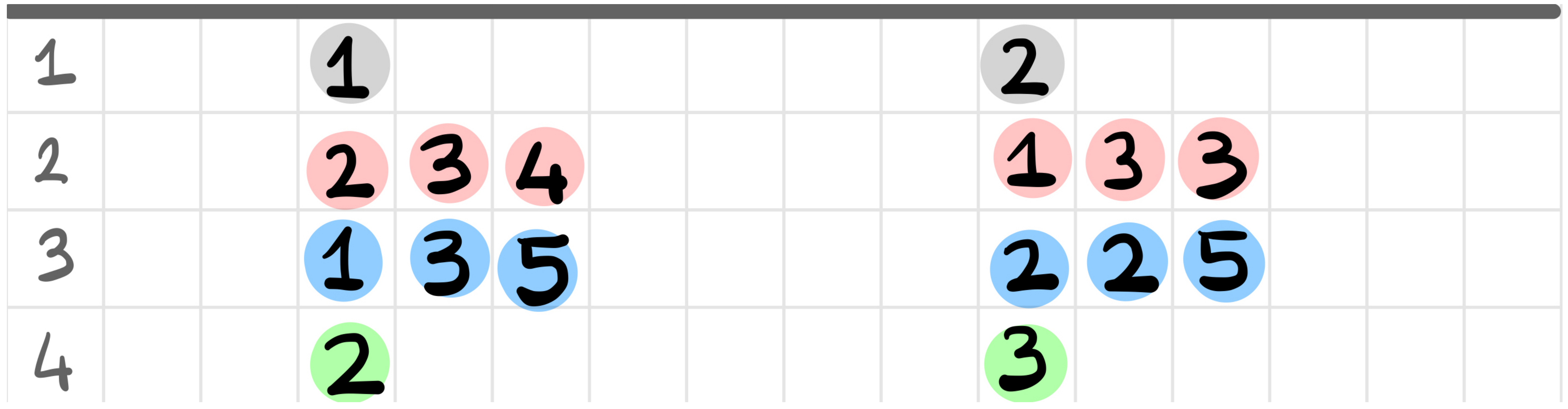


Construction of Υ

$$(M^k_{i,j}) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix} \longleftrightarrow ?$$

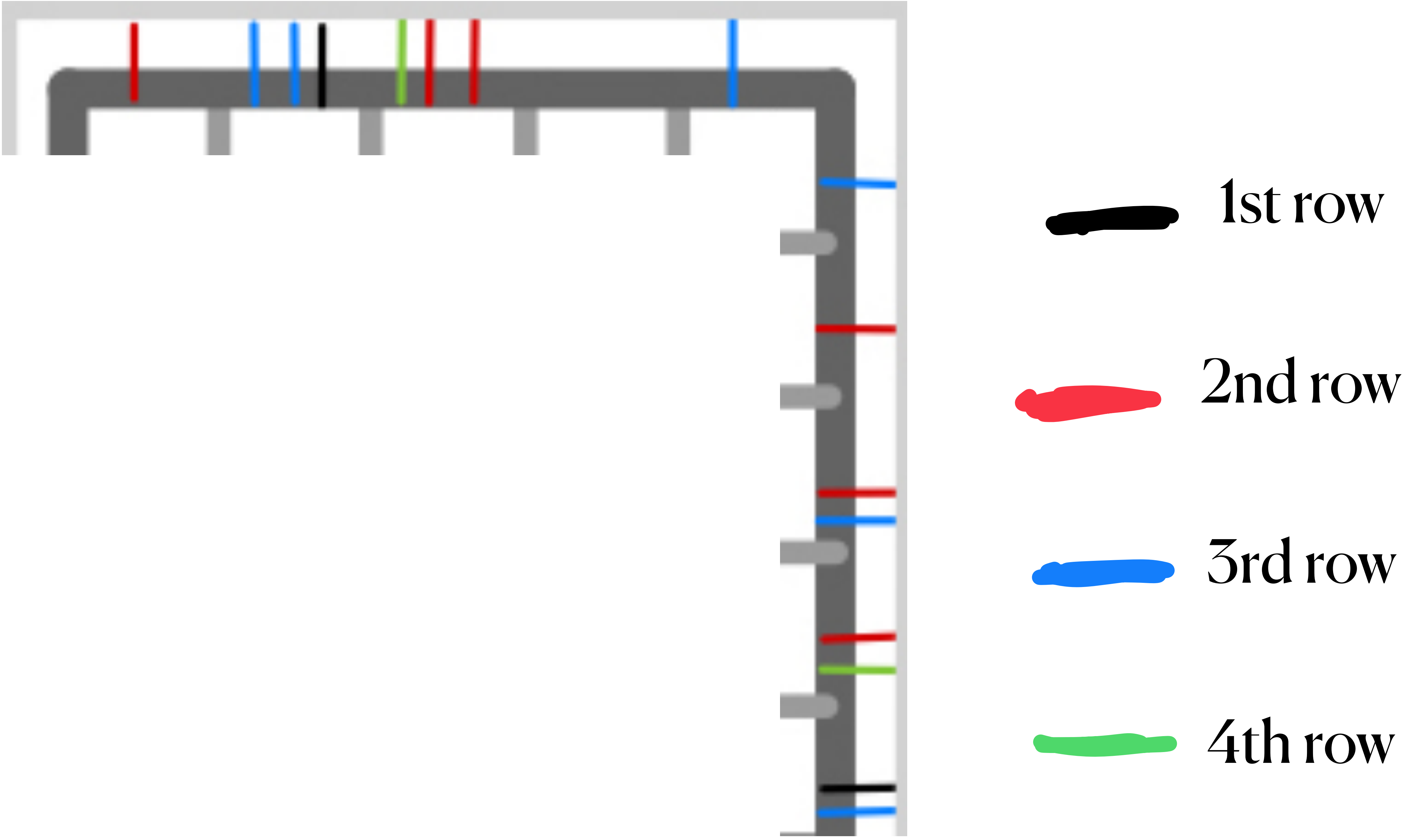


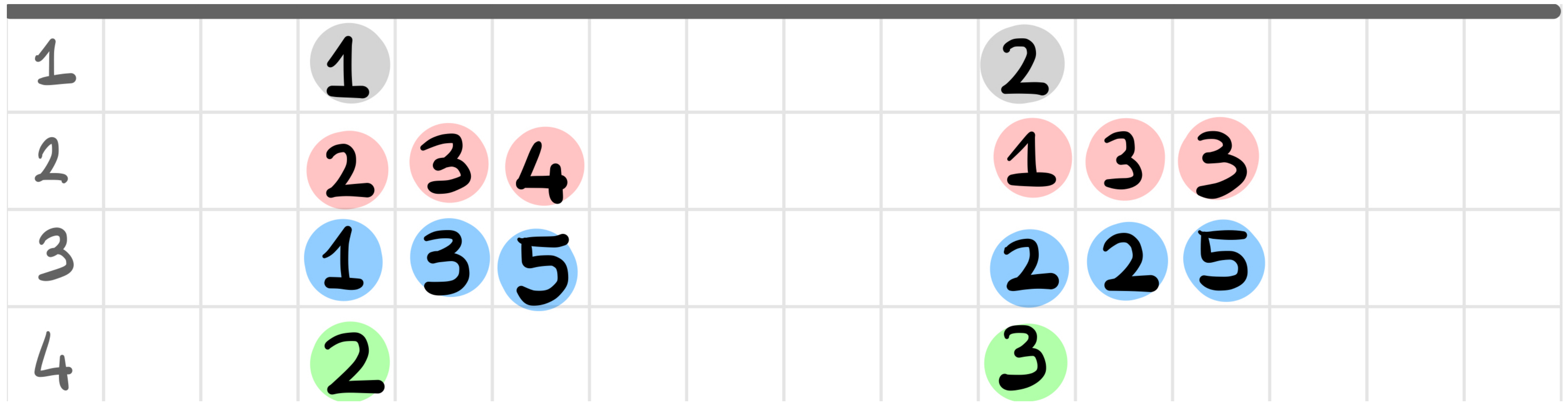


Construction of Υ

$$(M^k_{i,j}) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix} \longleftrightarrow ?$$

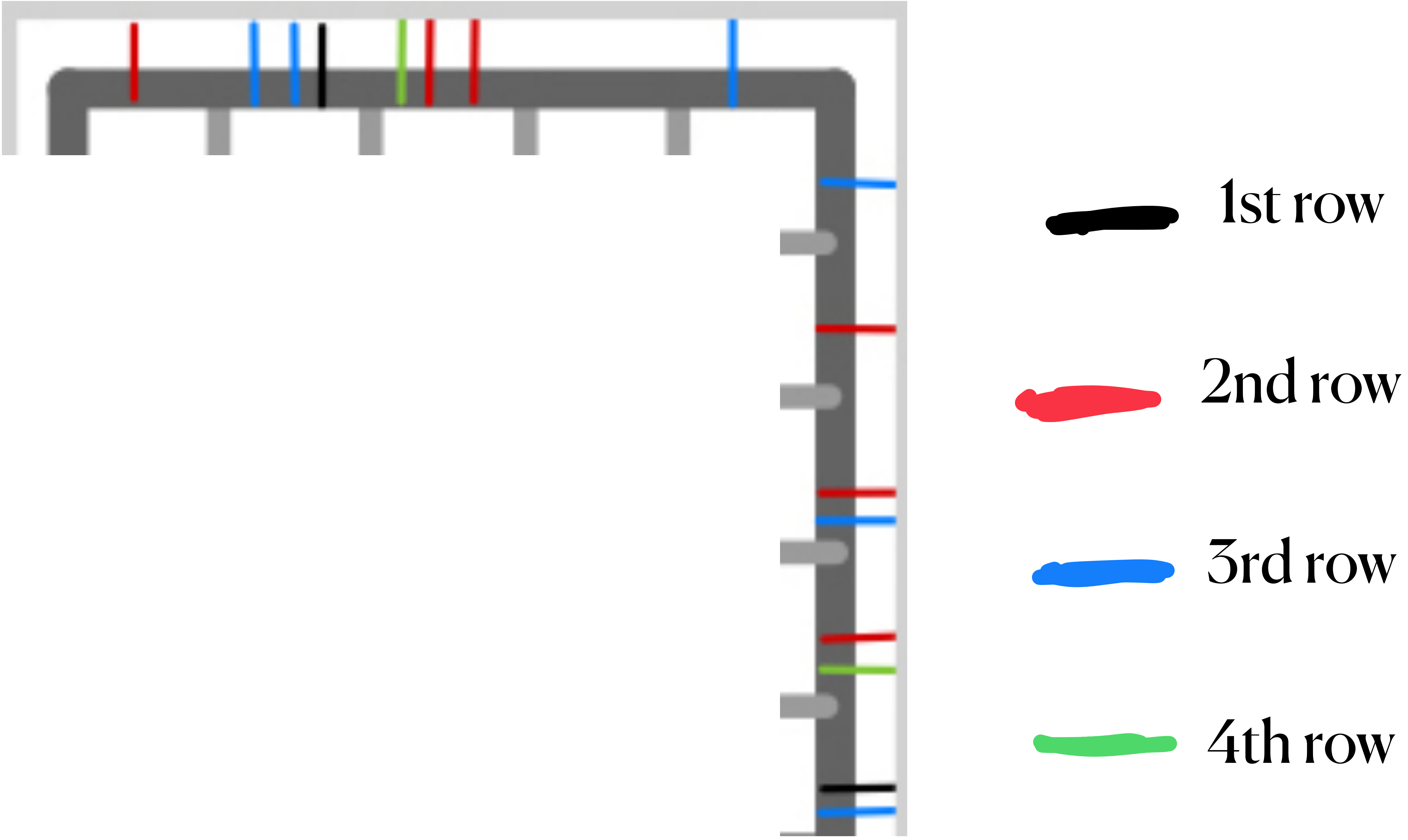


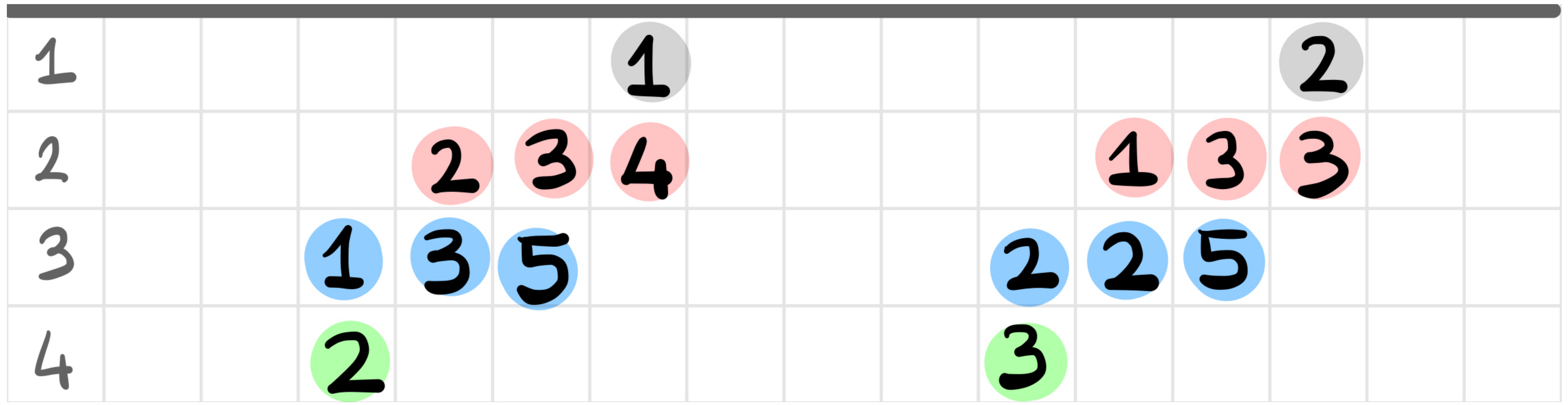


Produce a pair of semi-standard skew tableaux

Construction of Υ

$$(M^k_{i,j}) \longleftrightarrow (V, W; \kappa) \qquad \begin{pmatrix} (0,0,\ldots) & (0,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,1,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \end{pmatrix} \longleftrightarrow \ ?$$



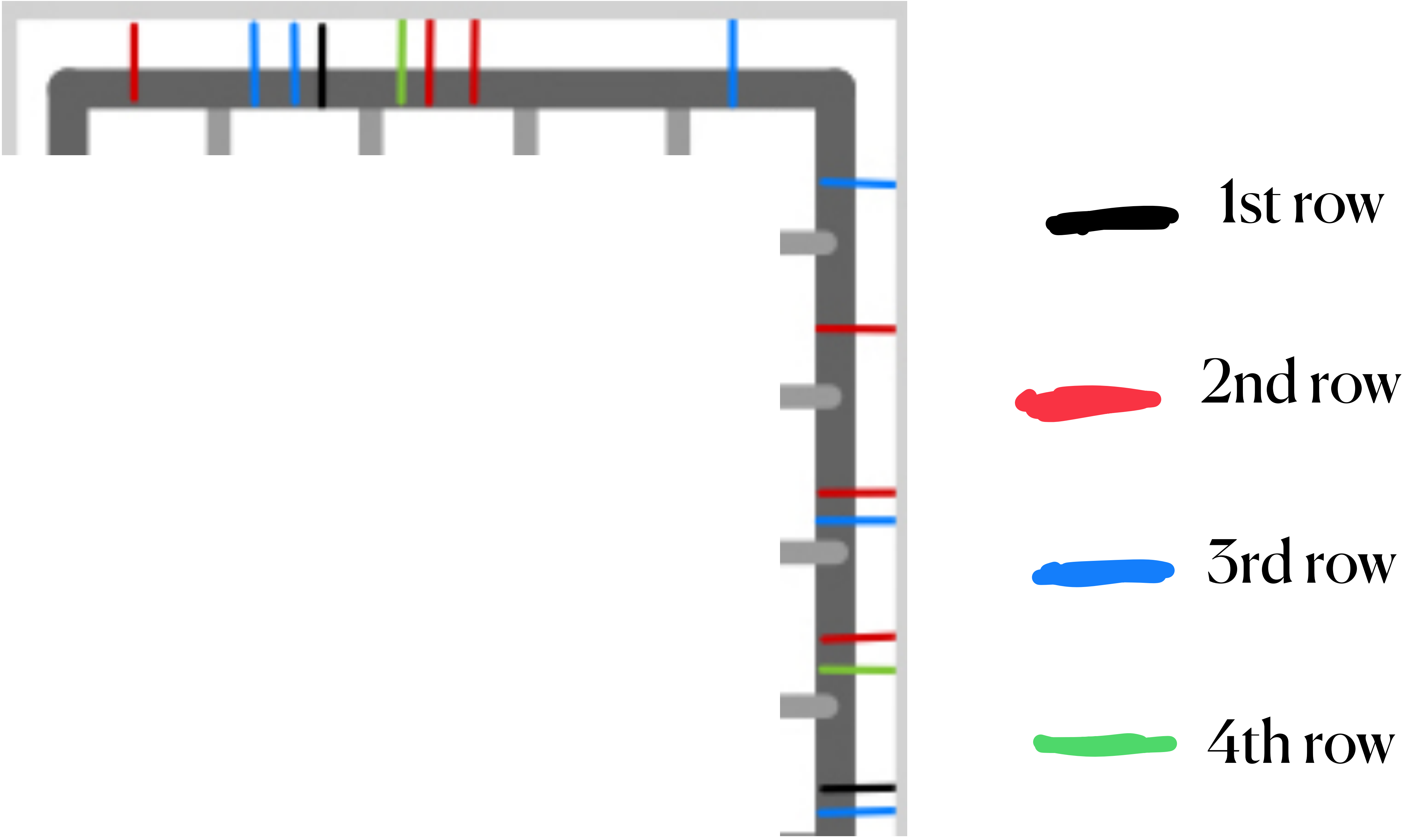


Produce a pair of semi-standard skew tableaux

Construction of Υ

$$(M_{i,j}^{k}) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\ldots) & (0,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,1,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \end{pmatrix} \longleftrightarrow \; ?$$



1						1							2		
2				2	3	4					1	3	3		
3			1	3	5					2	2	5			
4			2							3					

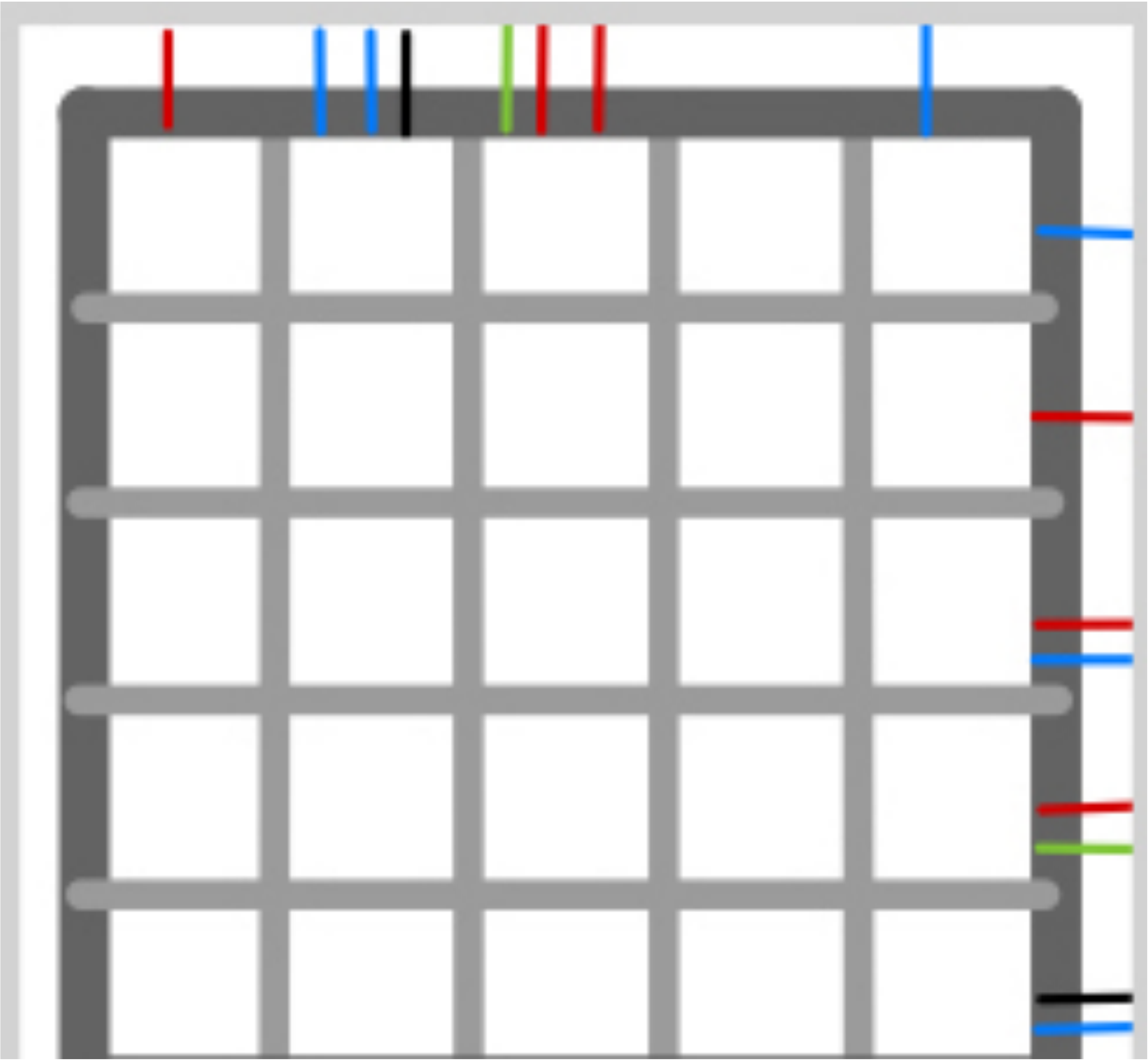
$P \quad , \quad Q$

Produce a pair of semi-standard skew tableaux

Construction of Υ

$$(M^k_{i,j}) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\ldots) & (0,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,1,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \end{pmatrix} \longleftrightarrow ?$$



1						1							2		
2				2	3	4					1	3	3		
3			1	3	5					2	2	5			
4			2							3					

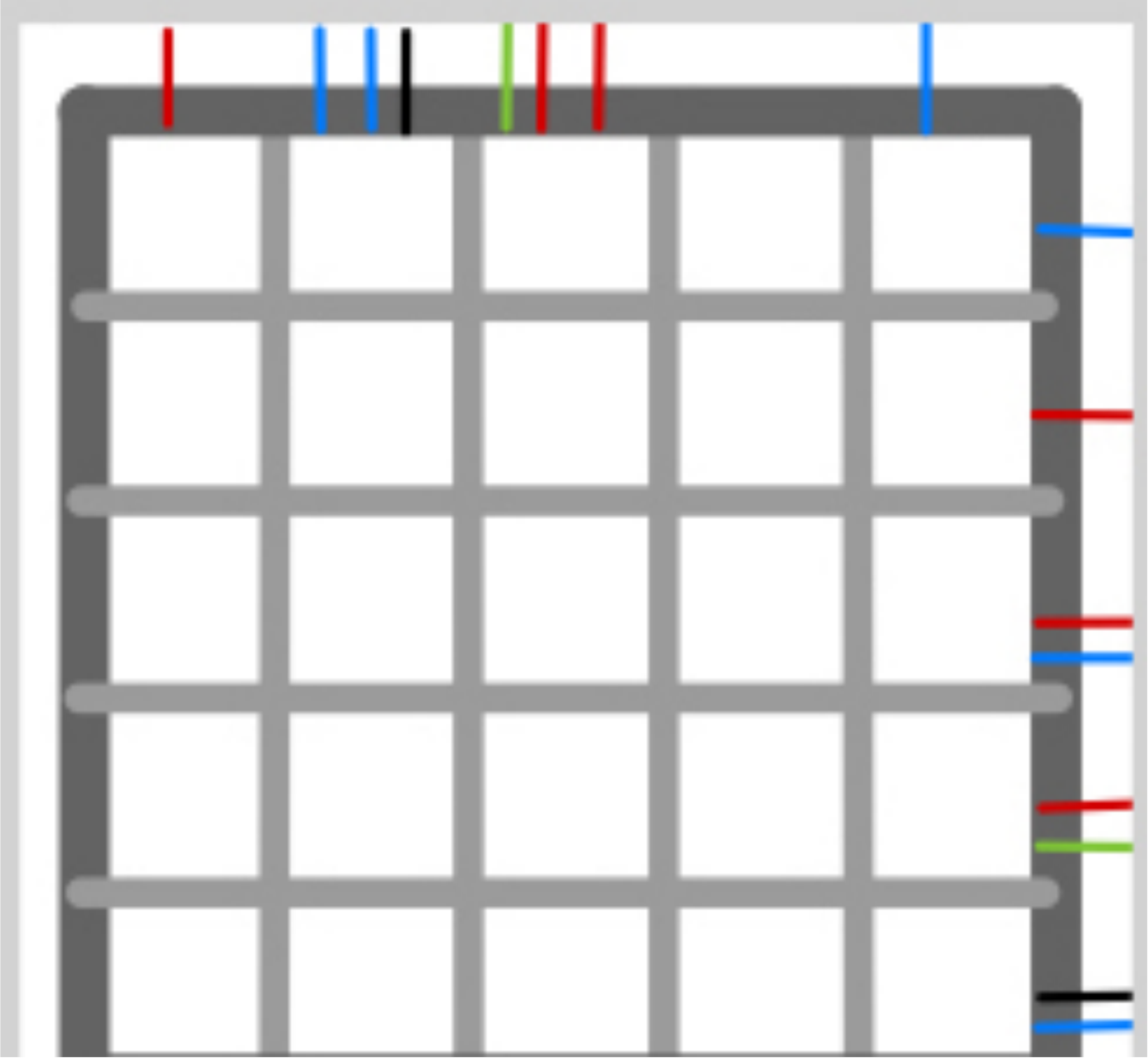
P , Q

Fact: if ρ = empty shape of P, Q, then $|\rho| = \sum_{k>0} \sum_{i,j=1}^{n} k\, M^k_{i,j}$ [Sagan-Stanley'89]

Construction of Υ

$$(M^k_{i,j}) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\ldots) & (0,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,1,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \end{pmatrix} \longleftrightarrow \; ?$$



1						1						2			
2				2	3	4				1	3	3			
3			1	3	5				2	2	5				
4			2						3						

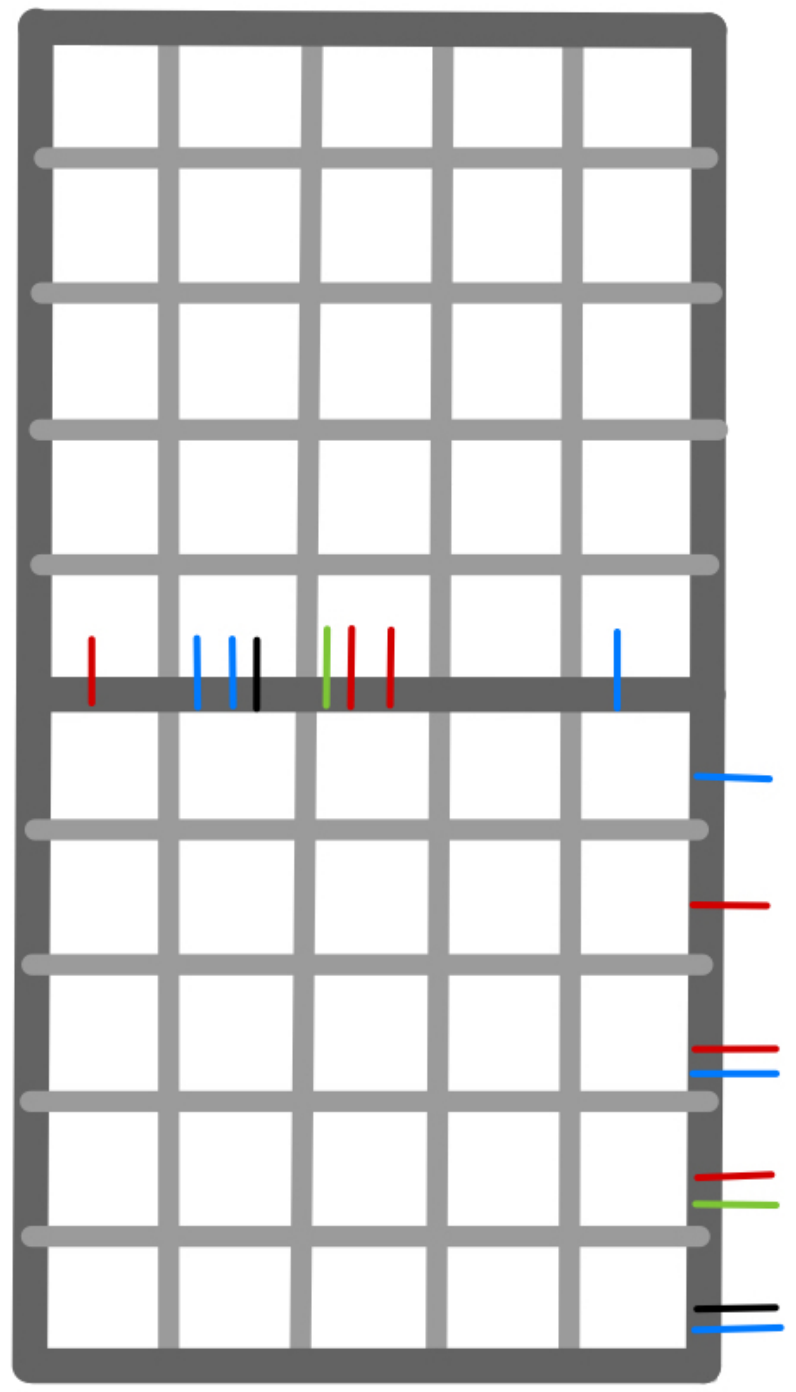
$P \quad , \quad Q$

IDEA: we think of (P, Q) as the initial data of an "integrable" dynamics

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\ldots) & (0,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,1,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \end{pmatrix} \longleftrightarrow \quad ?$$



1						1							2		
2				2	3	4					1	3	3		
3			1	3	5					2	2	5			
4			2							3					

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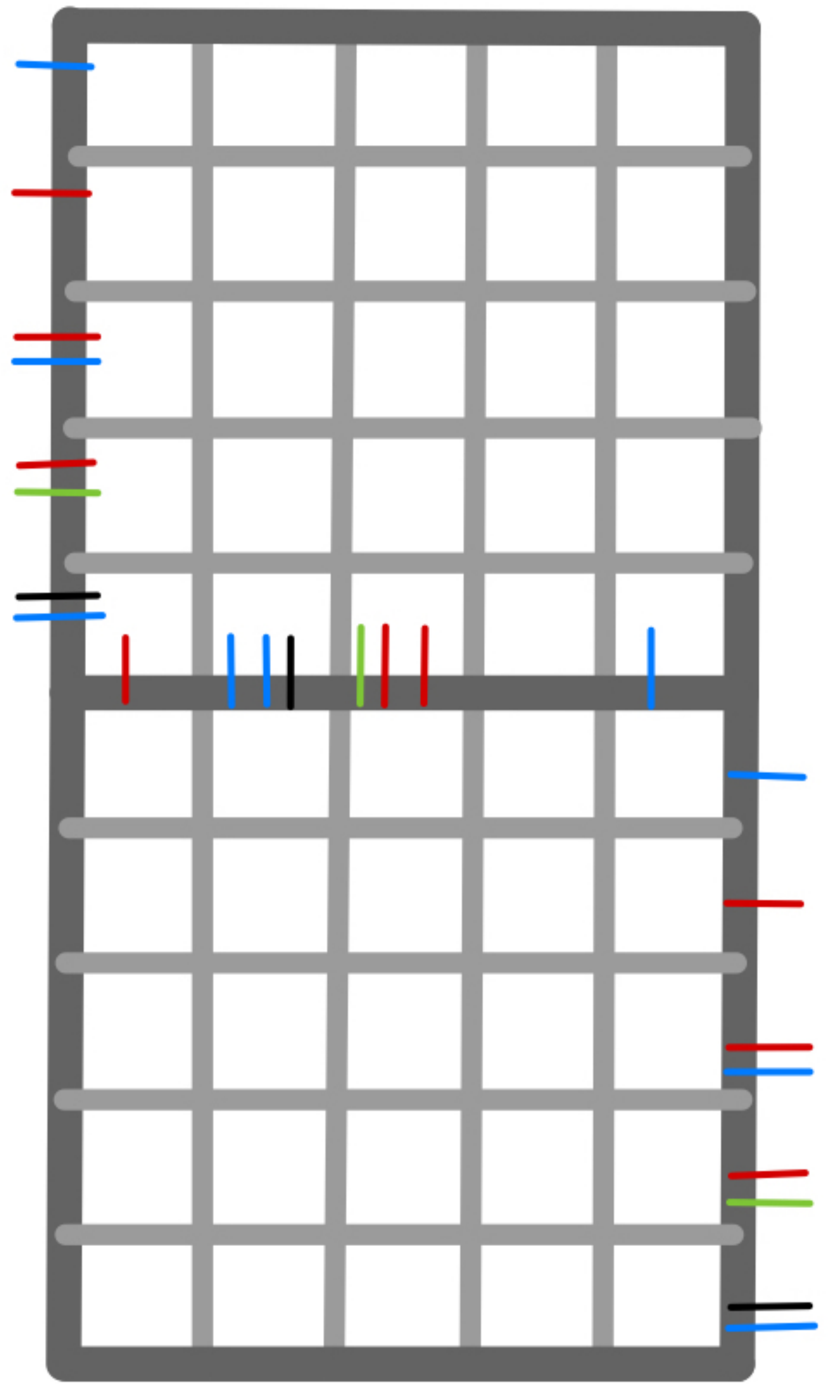

$$\begin{pmatrix} (0,0,\ldots) & (0,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,1,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \end{pmatrix} \longleftrightarrow \ ?$$

1						1							2		
2				2	3	4					1	3	3		
3			1	3	5					2	2	5			
4			2							3					

$P \quad , \quad Q$

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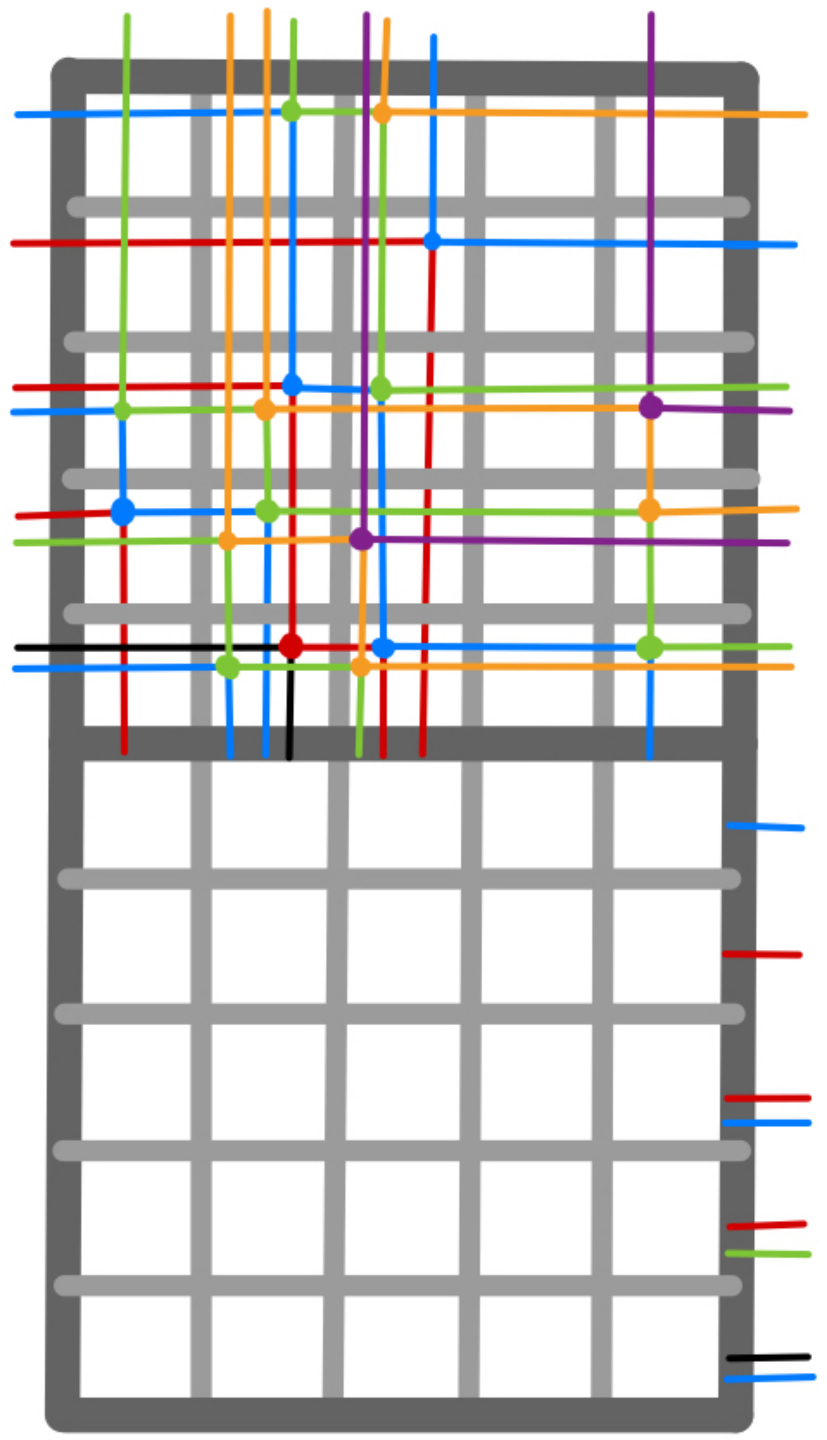
Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$



$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix} \longleftrightarrow \; ?$$

1						1							2		
2				2	3	4					1	3	3		
3			1	3	5					2	2	5			
4			2							3					

$$P \quad , \quad Q$$

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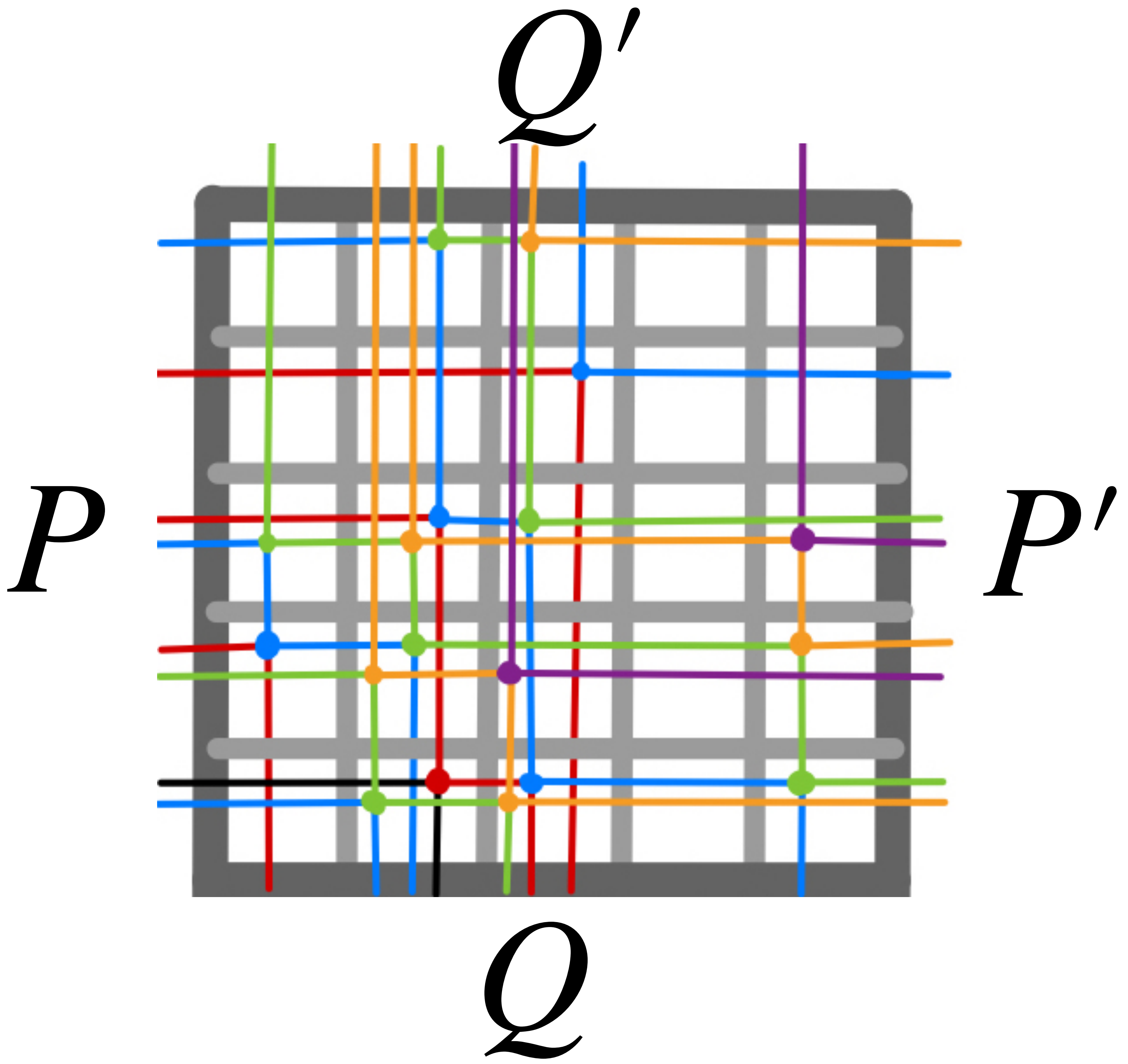
Construction of Υ

$$(M^k_{i,j}) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\ldots) & (0,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,1,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \end{pmatrix} \longleftrightarrow \; ?$$



1						1							2		
2				2	3	4					1	3	3		
3			1	3	5					2	2	5			
4			2							3					

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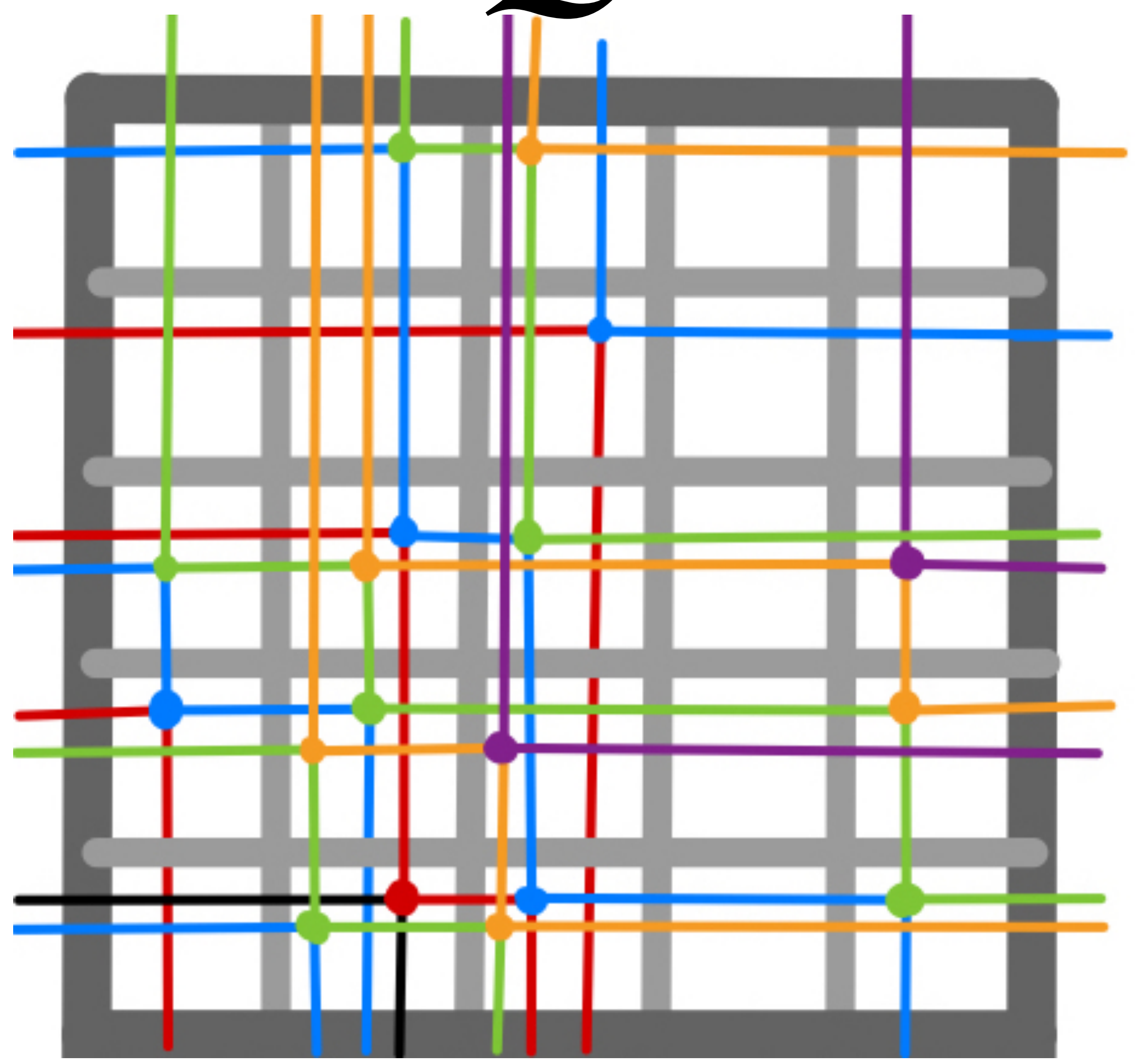
Construction of Υ

$$(M^k_{i,j}) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix} \longleftrightarrow \ ?$$

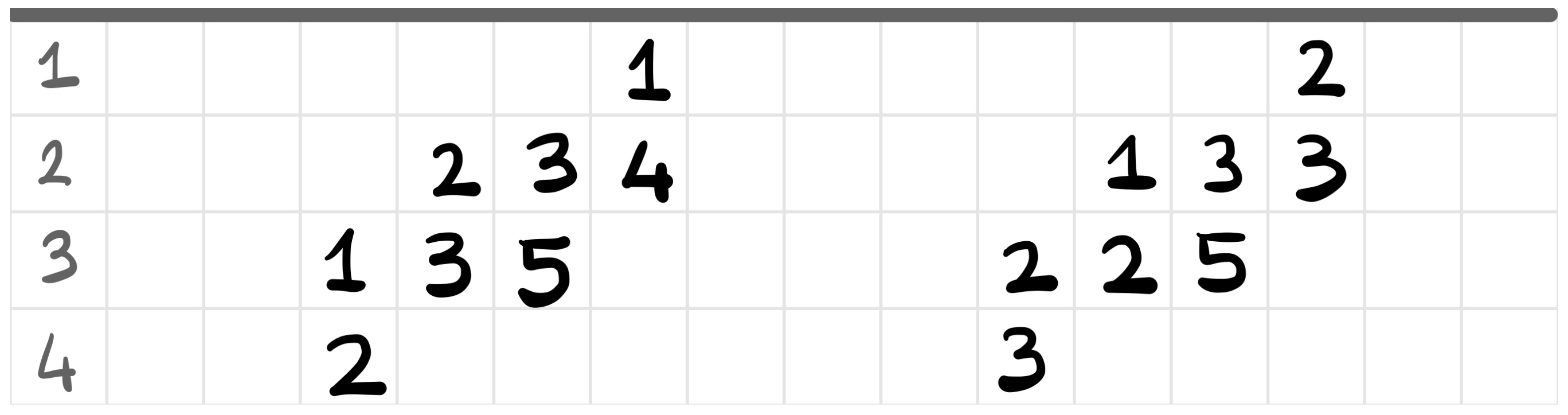
Q'

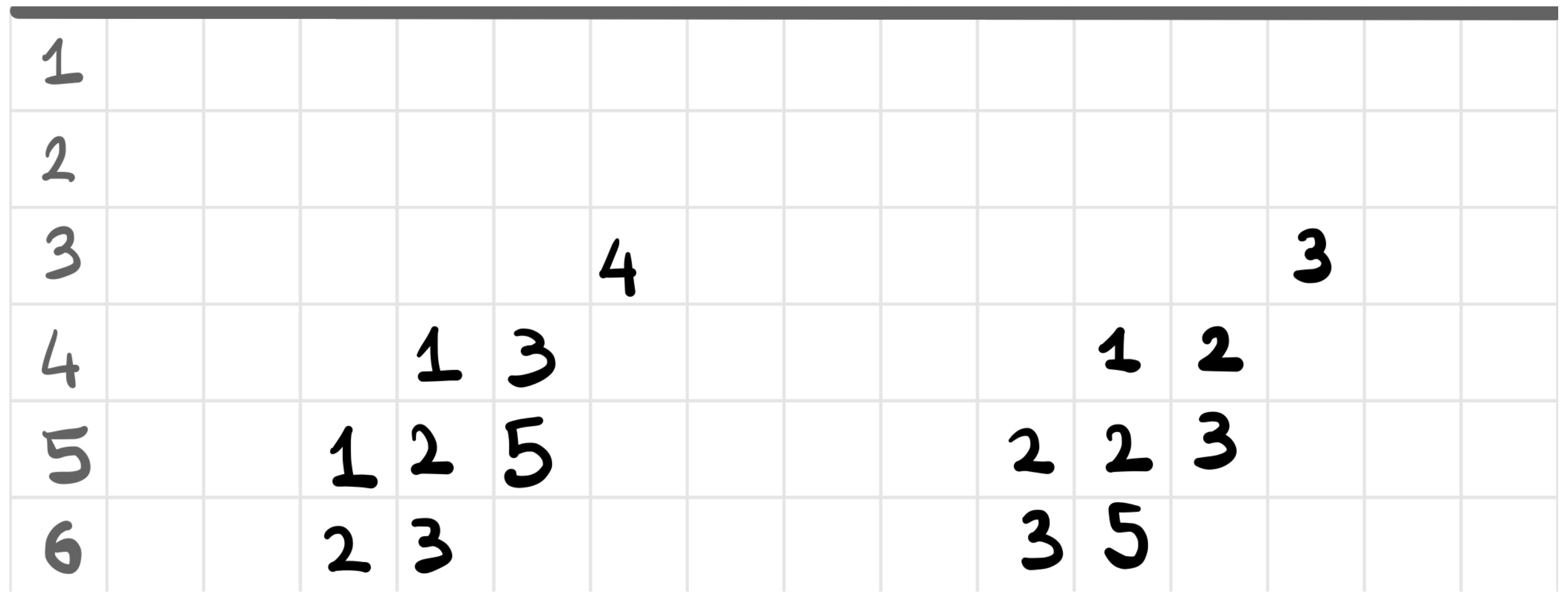
P



P'

Q





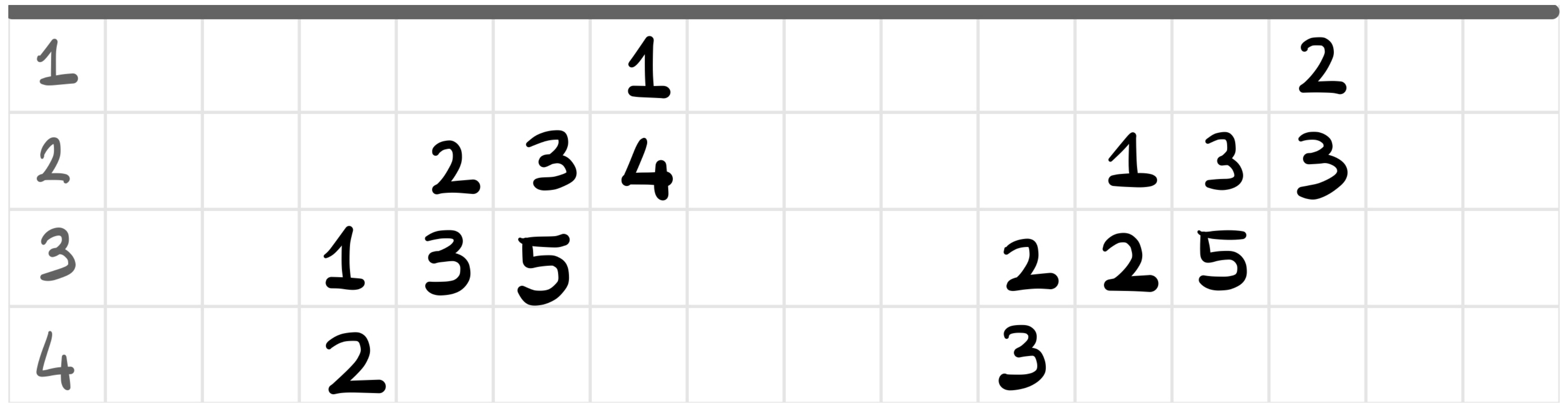
$P \quad , \quad Q \qquad\qquad P' \quad , \quad Q'$

IDEA: we think of (P, Q) as the initial data of an “integrable” dynamics

Construction of Υ

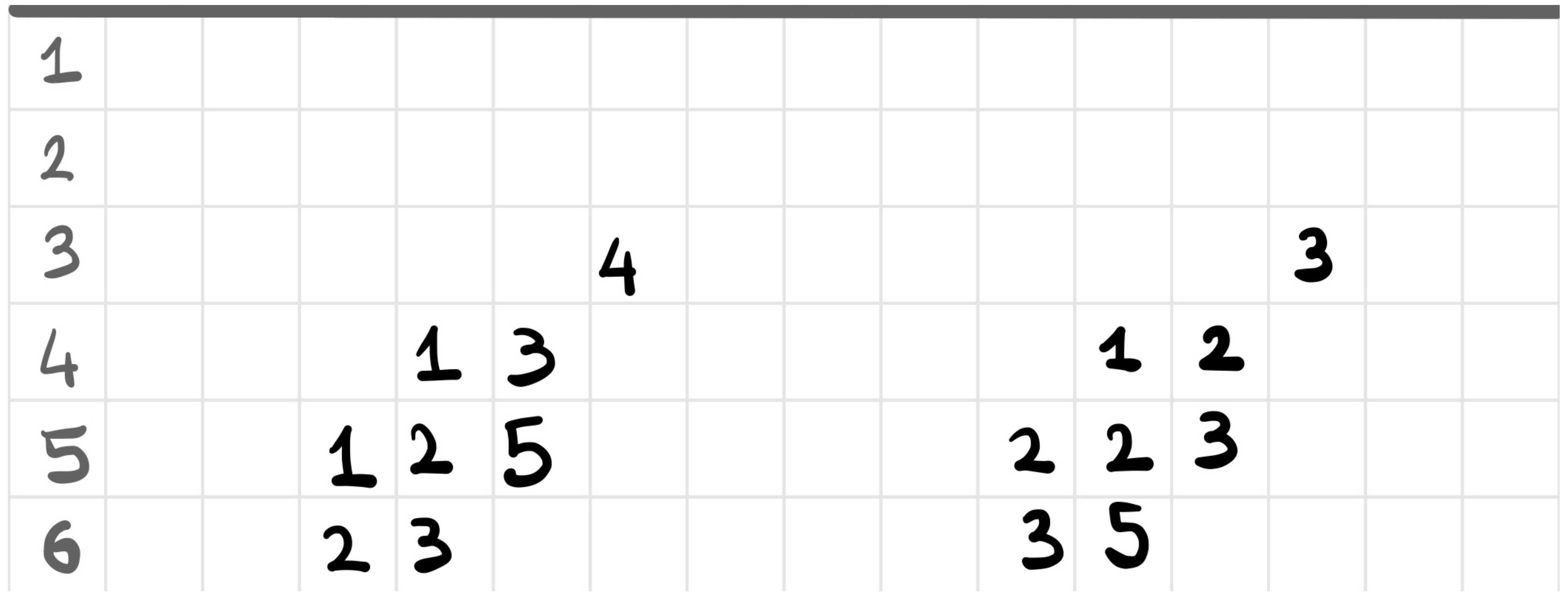
$$(M^k_{i,j}) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix} \longleftrightarrow \ ?$$



$P \quad , \quad Q$

RSK $\longrightarrow$



$P' \quad , \quad Q'$

IDEA: we think of (P, Q) as the initial data of an "integrable" dynamics

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix} \longleftrightarrow \quad ?$$

$$(P, Q) \overset{\textbf{RSK}}{\to} (P', Q')$$

IDEA: we think of (P, Q) as the initial data of an "integrable" dynamics

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa) \qquad \begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix} \longleftrightarrow \quad ?$$

$$(P, Q) \overset{\textbf{RSK}}{\to} (P', Q') \to \cdots \to (P^{(n)}, Q^{(n)})$$

Skew **RSK** dynamics

IDEA: we think of (P, Q) as the initial data of an "integrable" dynamics

Construction of Υ

$$(M^k_{i,j}) \longleftrightarrow (V, W; \kappa) \qquad \begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix} \longleftrightarrow \quad ?$$

$$(P, Q) \overset{\mathbf{RSK}}{\to} (P', Q') \to \cdots \to (P^{(n)}, Q^{(n)})$$

Skew **RSK** dynamics

Fact: Asymptotically the tableaux $P^{(n)}, Q^{(n)}$ become “stable”

IDEA: we think of (P, Q) as the initial data of an “integrable” dynamics

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\dots) & (0,0,\dots) & (0,1,\dots) & (0,0,\dots) & (1,0,\dots) \\ (1,0,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,1,\dots) & (0,1,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) \\ (0,0,\dots) & (1,0,\dots) & (0,0,\dots) & (0,0,\dots) & (0,0,\dots) \end{pmatrix} \longleftrightarrow \quad ?$$

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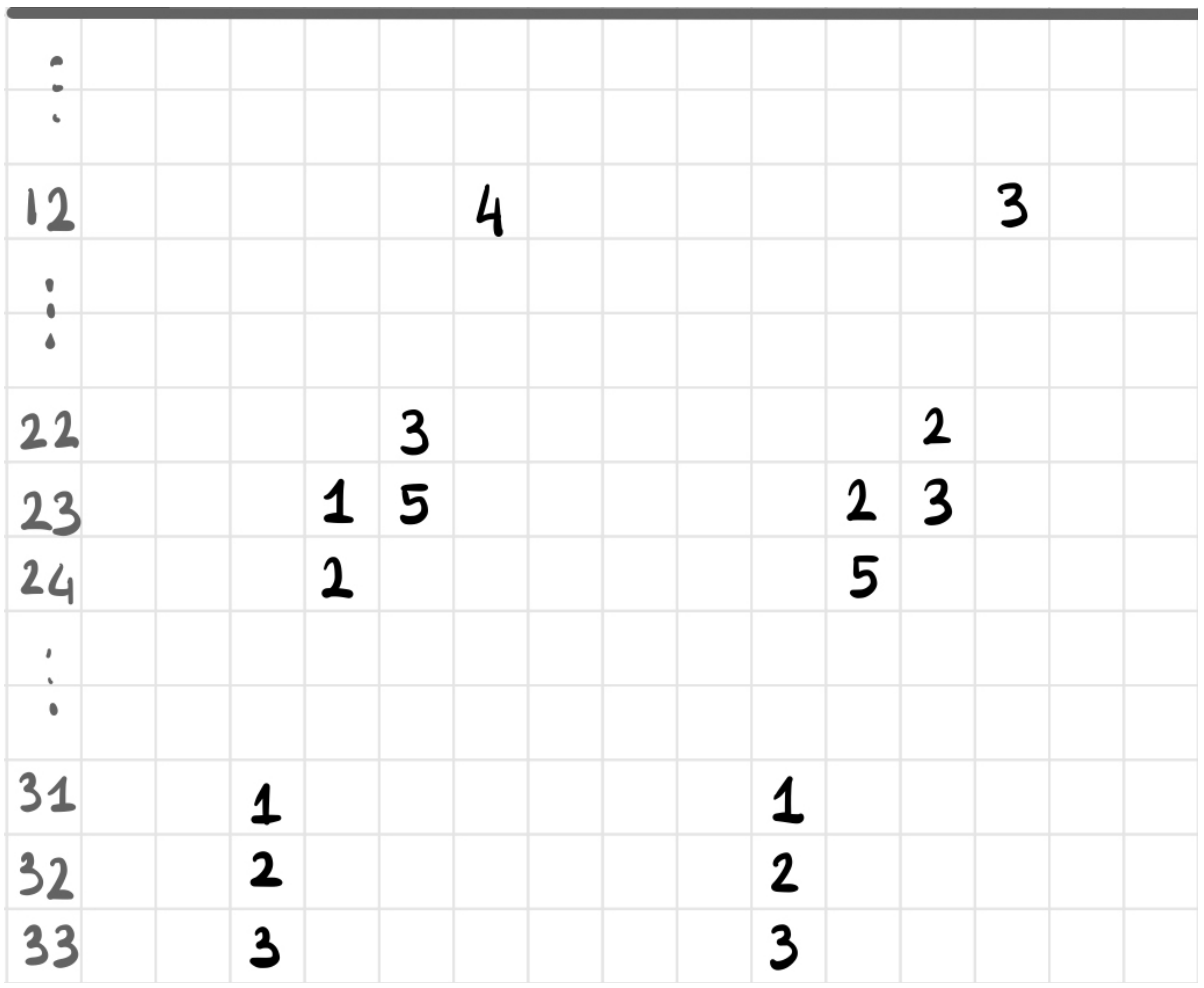


$P^{(10)}$, $Q^{(10)}$

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

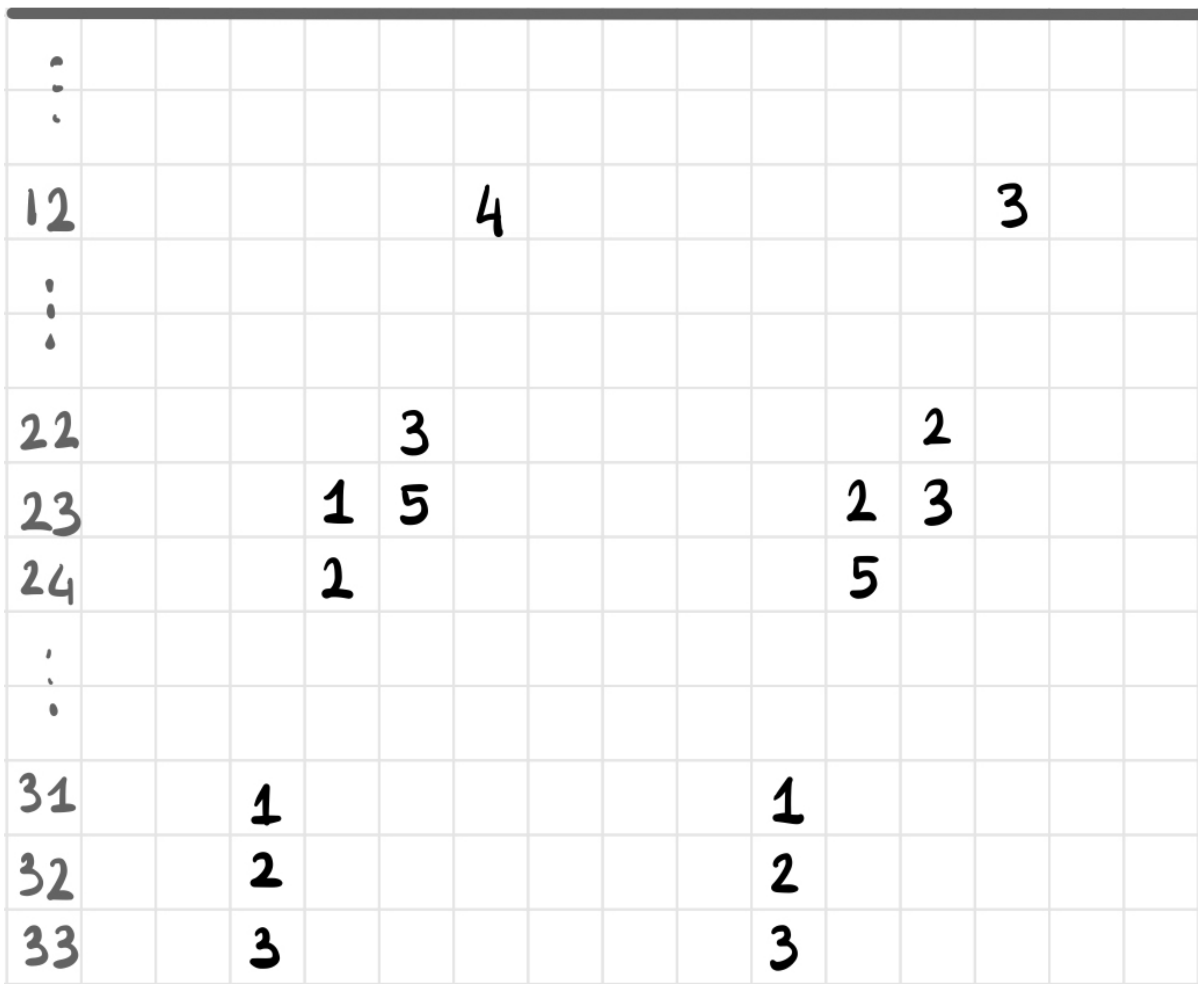
$$\begin{pmatrix} (0,0,\ldots) & (0,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,1,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \end{pmatrix} \longleftrightarrow \quad ?$$

$$(P, Q) \overset{\textbf{RSK}}{\to} (P', Q') \to \cdots \to (P^{(n)}, Q^{(n)})$$

Skew **RSK** dynamics

Fact: Asymptotically the tableaux $P^{(n)}, Q^{(n)}$ become "stable"

From stable configurations we determine vertically strict tableaux V, W

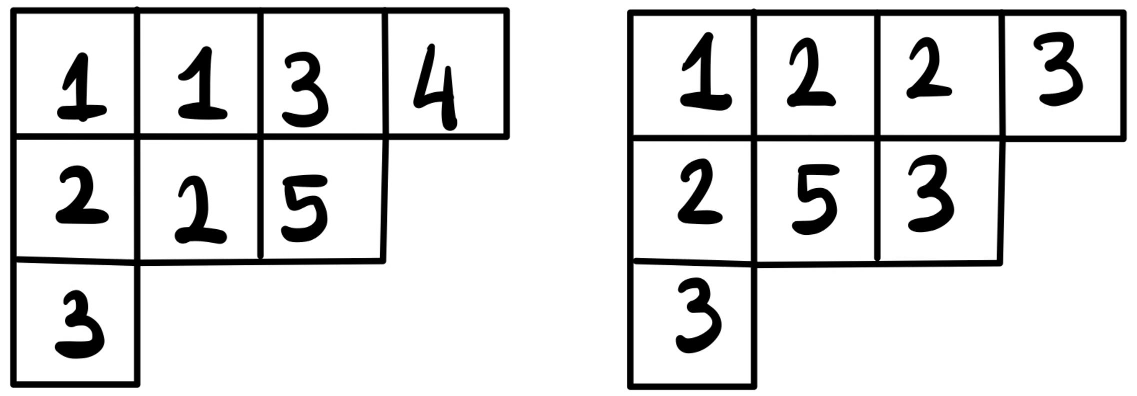


$P^{(10)}$, $Q^{(10)}$

Construction of Υ

$$(M_{i,j}^k) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\ldots) & (0,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,1,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \end{pmatrix} \longleftrightarrow$$



$; \kappa$

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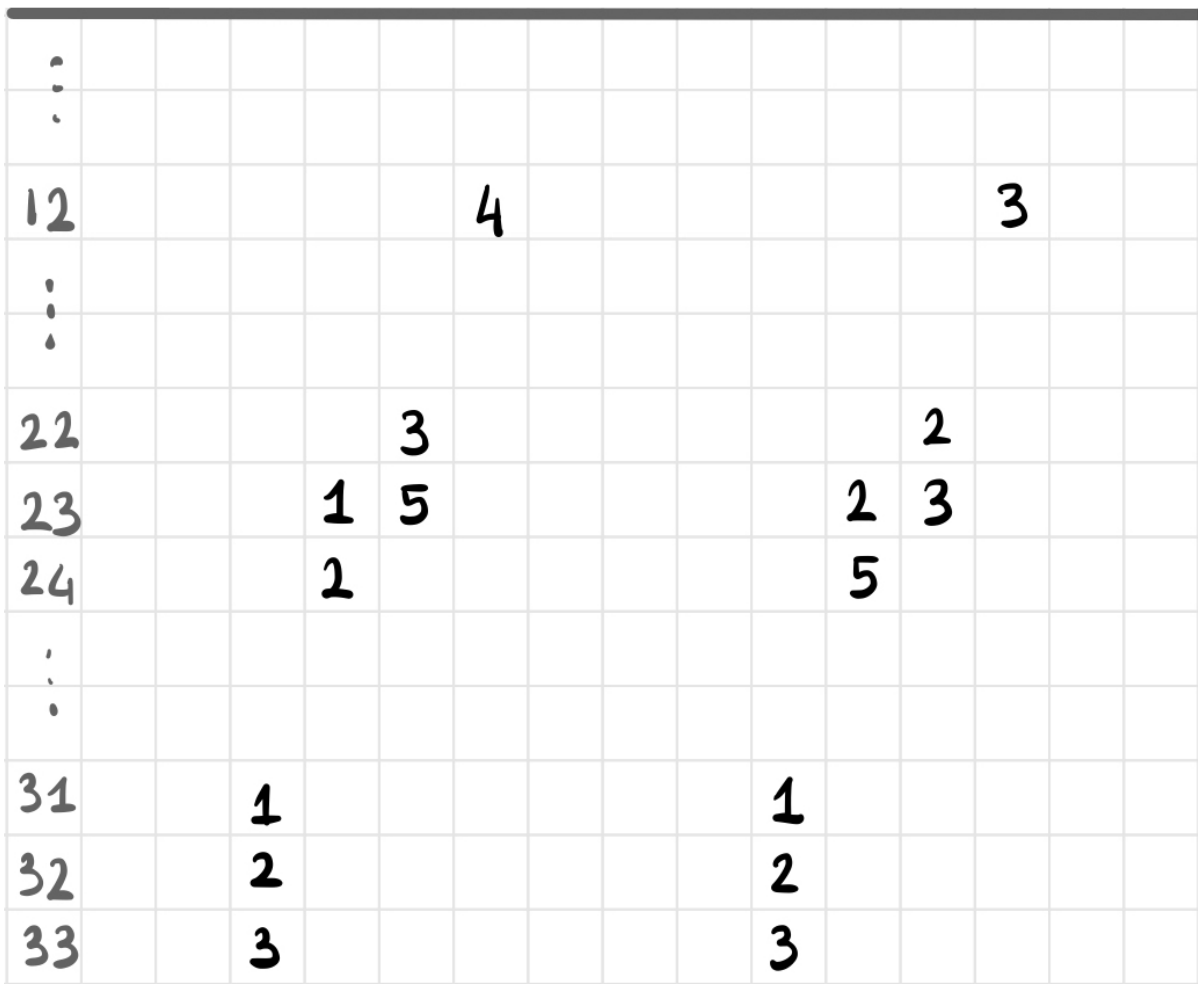
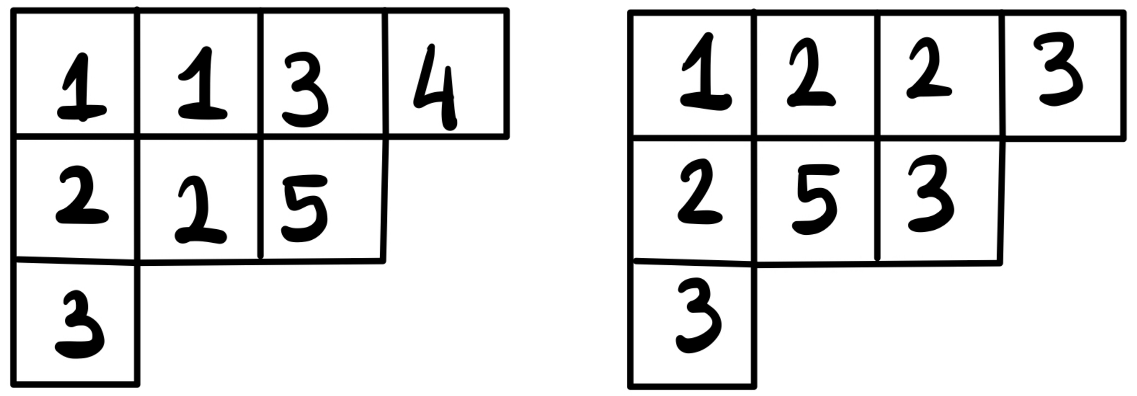


$P^{(10)}$, $Q^{(10)}$

Construction of Υ

$$(M^k_{i,j}) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\ldots) & (0,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,1,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \end{pmatrix} \longleftrightarrow$$



$; \kappa$

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Skew **RSK** dynamics

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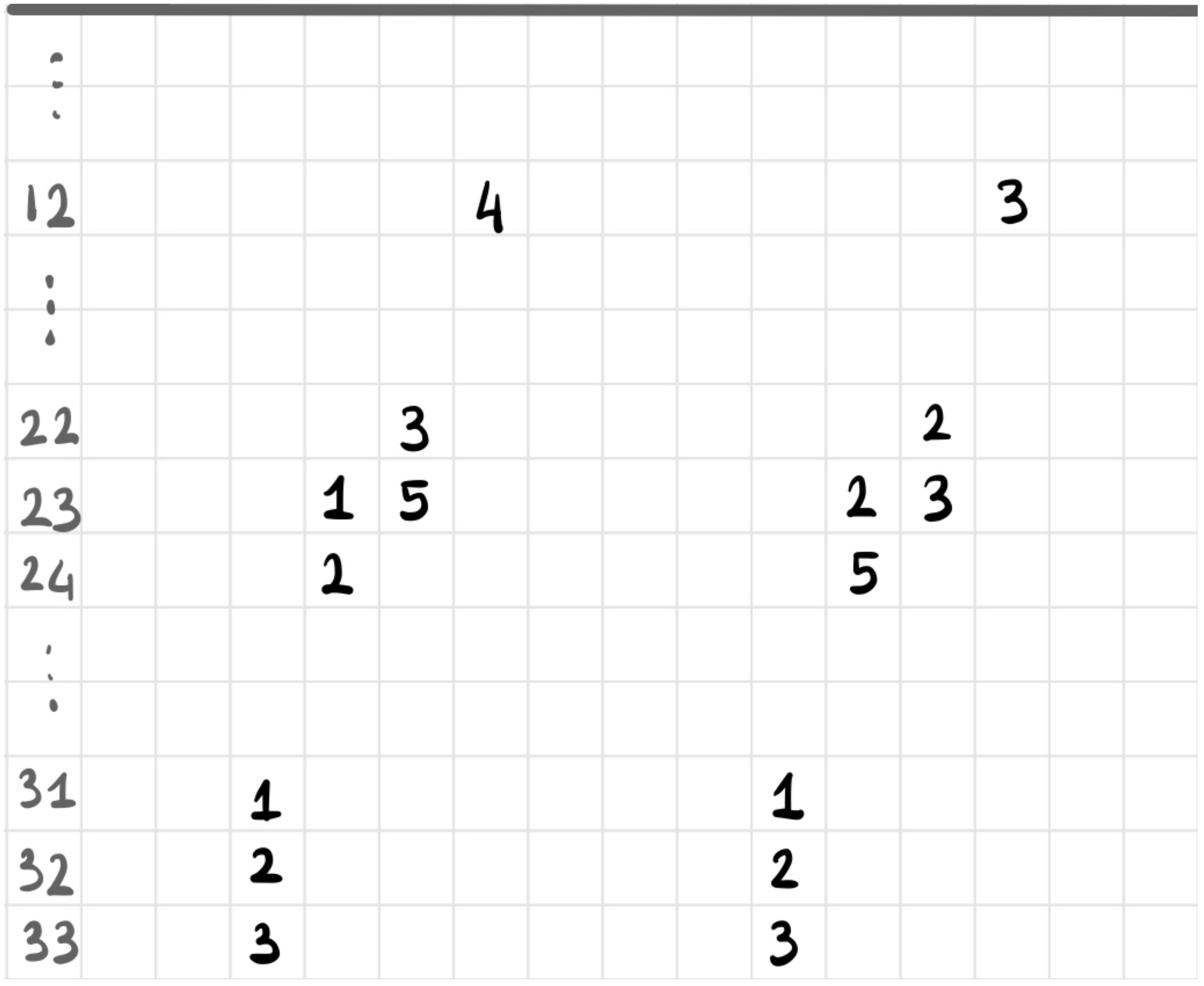
From stable configurations we determine vertically strict tableaux V, W

To determine κ we need to study the scattering of the skew RSK dynamics (no time today)

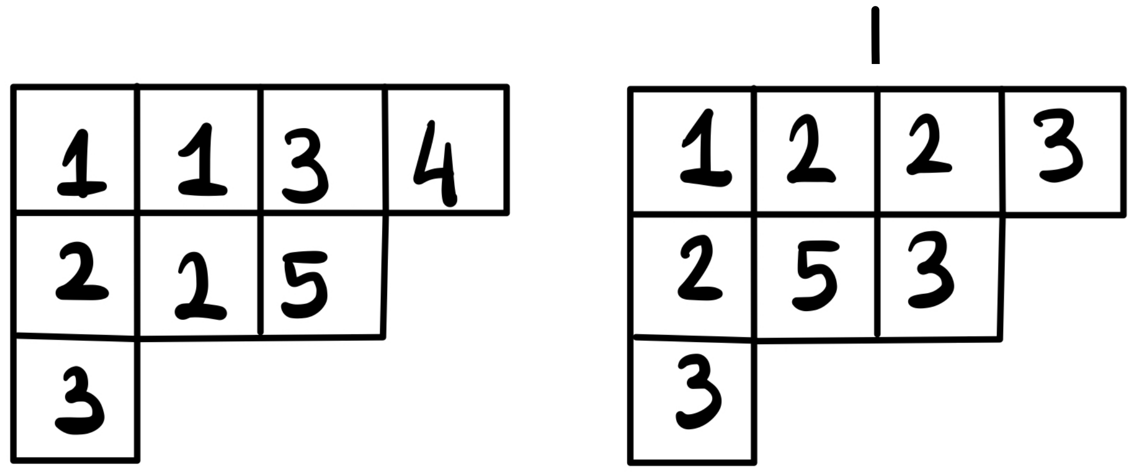


$P^{(10)}$, $Q^{(10)}$

Construction of Υ

$$(M^k_{i,j}) \longleftrightarrow (V, W; \kappa)$$

$$\begin{pmatrix} (0,0,\ldots) & (0,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (1,0,\ldots) \\ (1,0,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,1,\ldots) & (0,1,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \\ (0,0,\ldots) & (1,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) & (0,0,\ldots) \end{pmatrix} \longleftrightarrow$$



$; \kappa$

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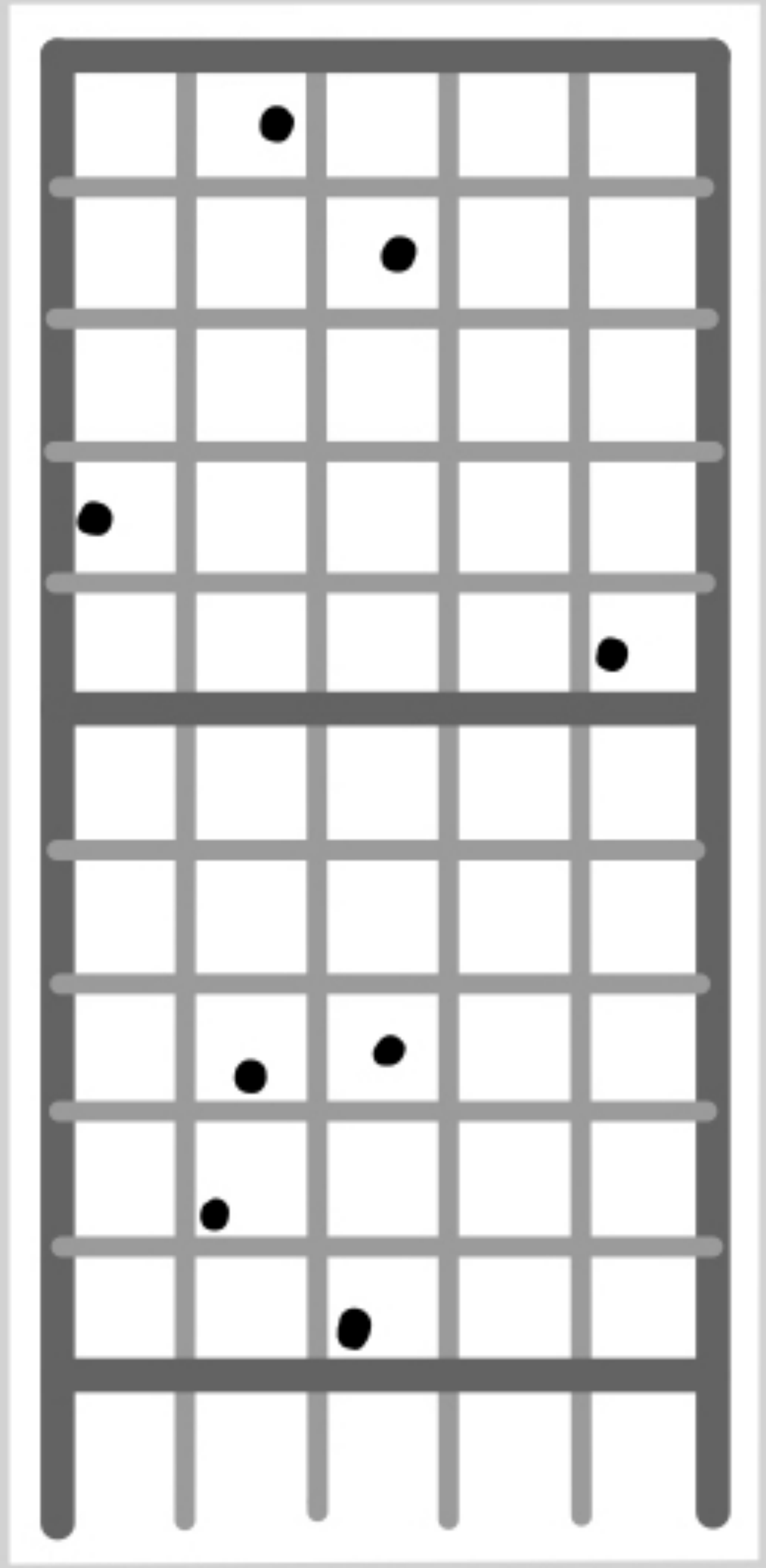
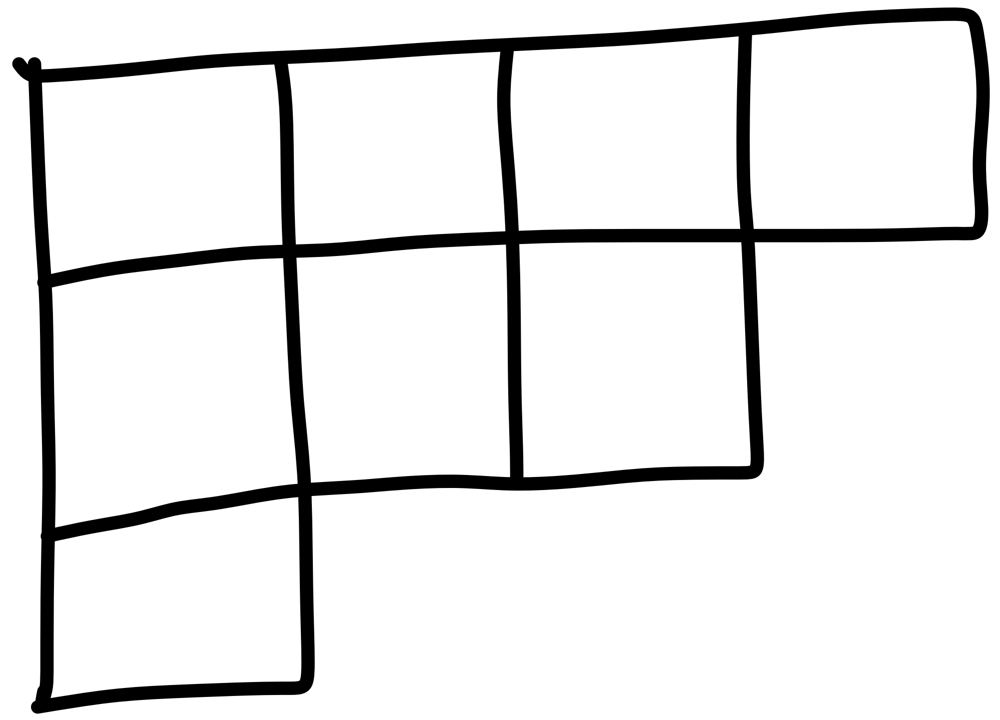
To determine κ we need to study the scattering of the skew RSK dynamics (no time today)

Question: Can we characterize the shape μ of V, W in terms of the matrix $M^k_{i,j}$?

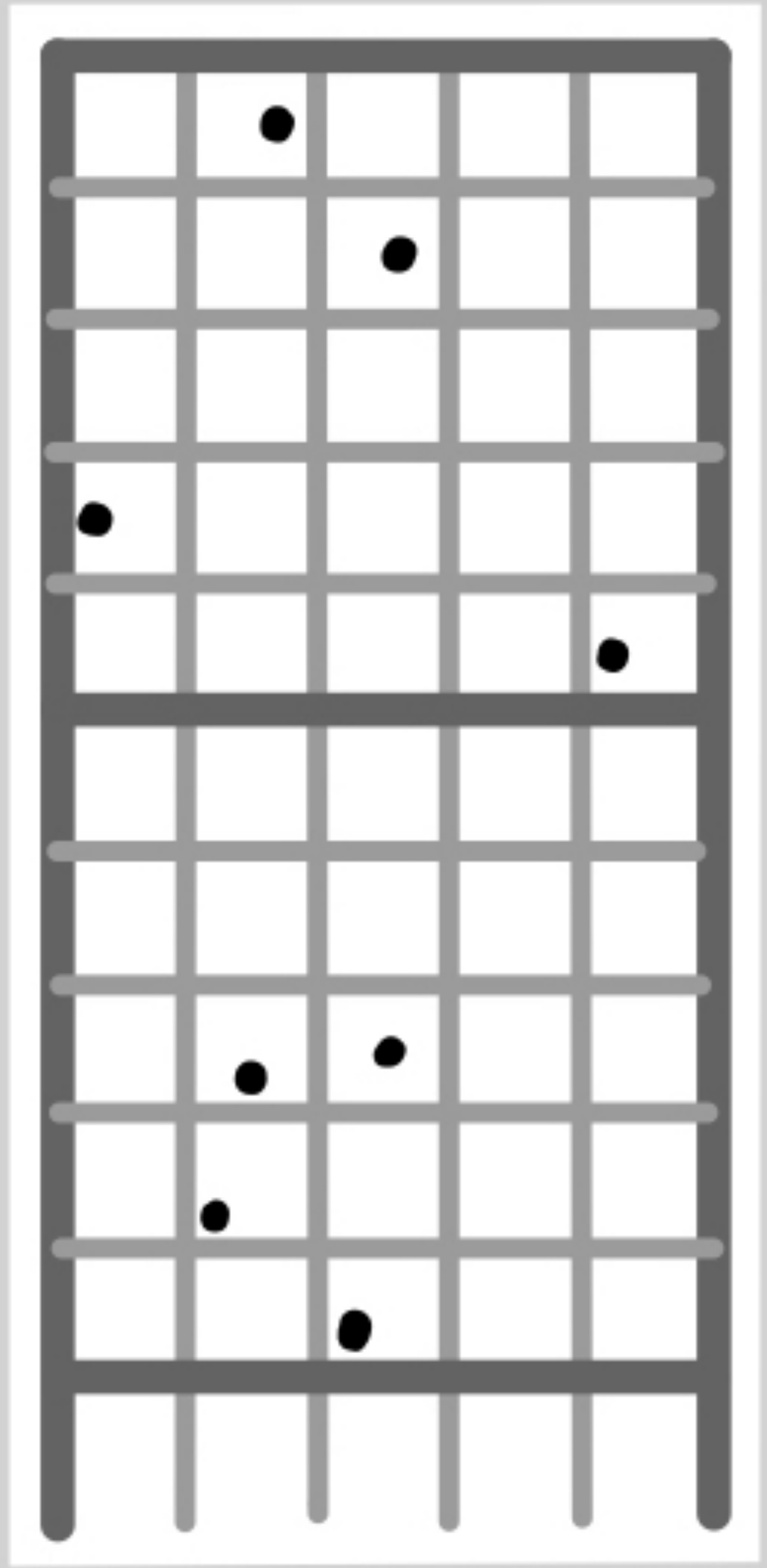
Greene invariants



- **Definition:**
  - **LDS**$_k$ = maximal number of bullets in k nonintersecting loops
  - **LIS**$_k$ = maximal number of bullets in k nonintersecting up-right paths



Greene invariants



- **Definition:**
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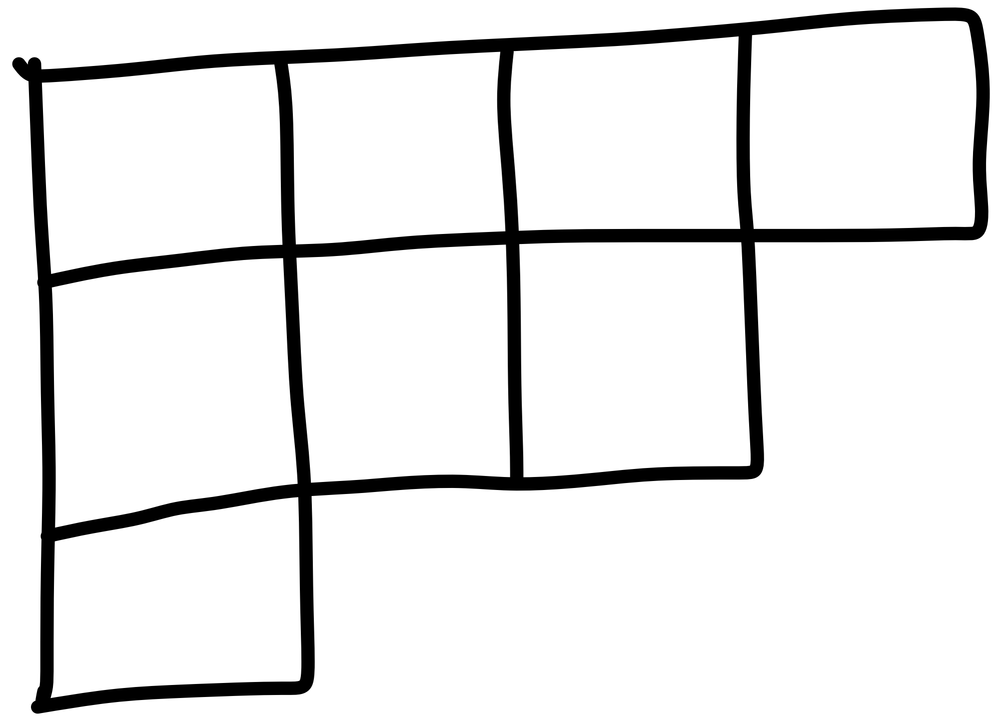
  - $\mathbf{LIS}_k$ = maximal number of bullets in k nonintersecting up-right paths



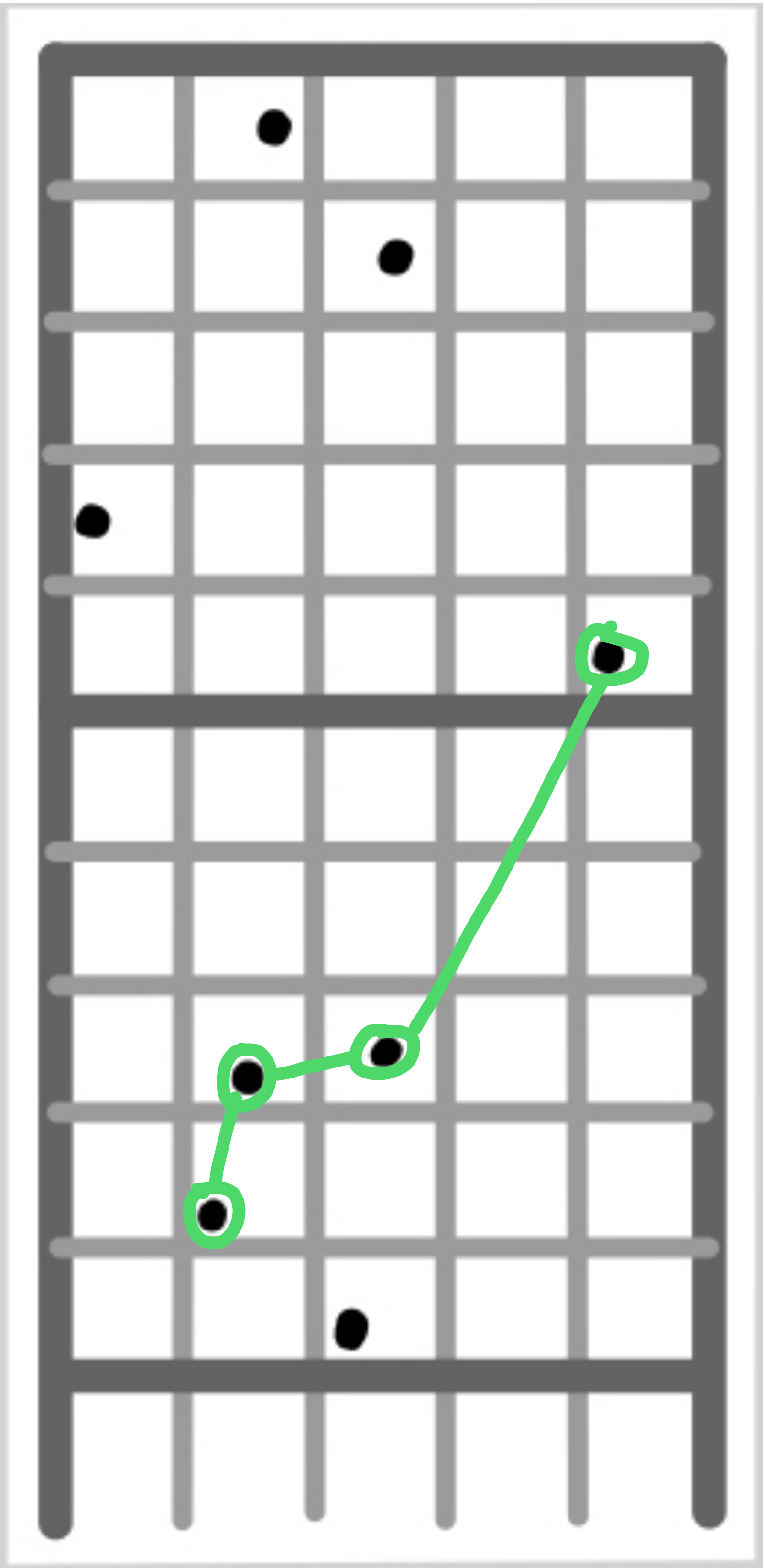
Theorem [IMS'21]

$\mathbf{LDS}_k$ and $\mathbf{LIS}_k$ determine the shape μ of (V, W)

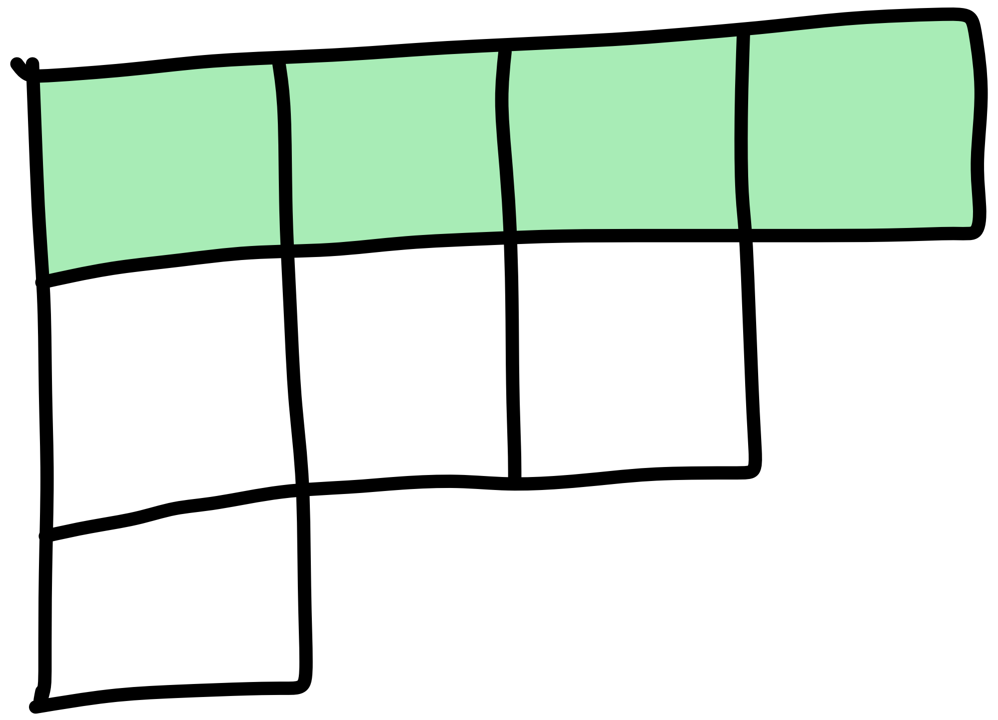
$$\mu_1 = \mathrm{LIS}_1 \qquad \mu_1' = \mathrm{LDS}_1$$

$$\mu_1 + \cdots + \mu_\ell = \mathrm{LIS}_\ell \qquad \mu_1' + \cdots + \mu_\ell' = \mathrm{LDS}_\ell$$

Greene invariants



- **Definition:**
  - **LDS**$_k$ = maximal number of bullets in k nonintersecting loops
  - **LIS**$_k$ = maximal number of bullets in k nonintersecting up-right paths



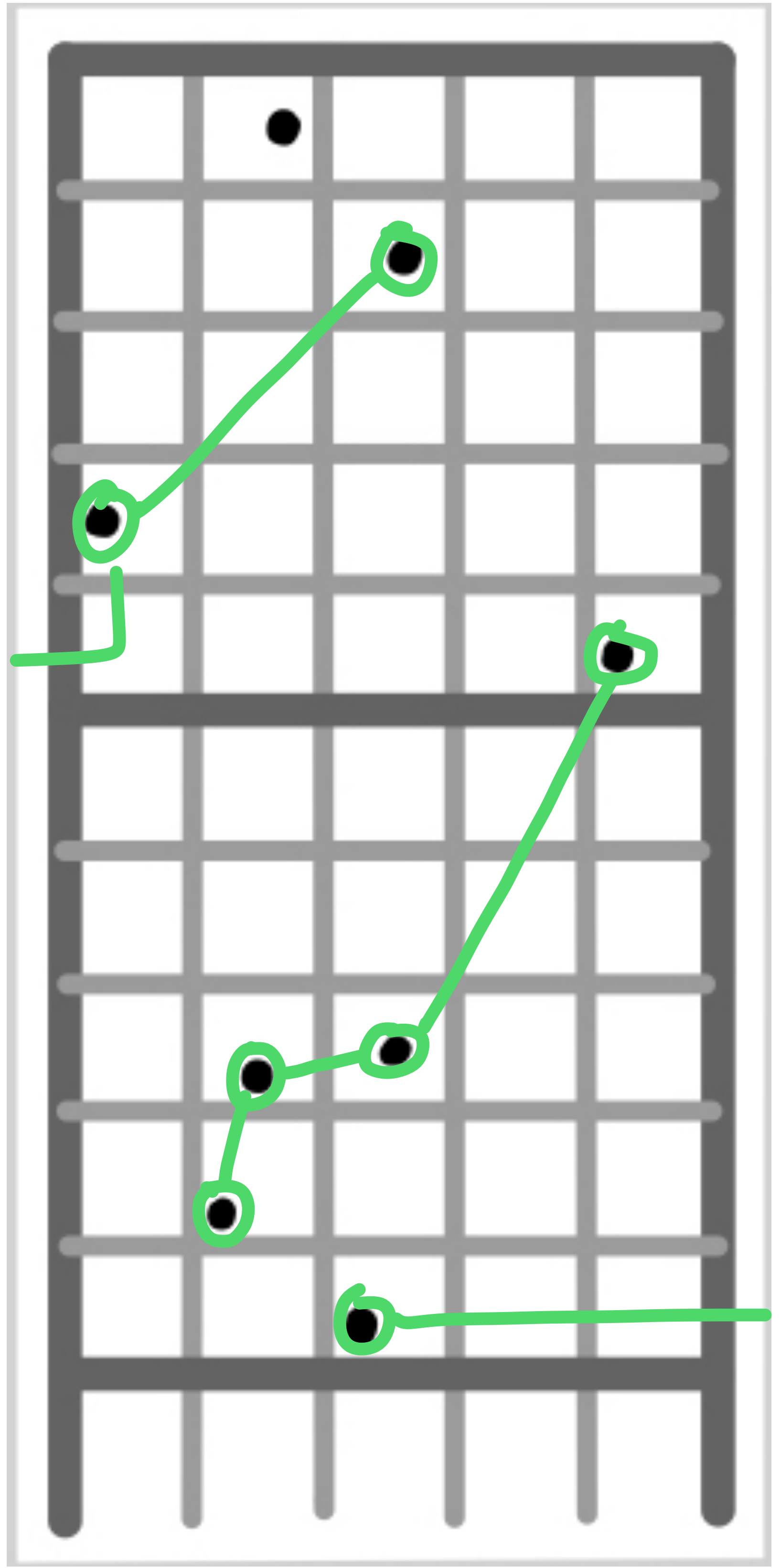
Theorem [IMS'21]

LDS$_k$ and **LIS**$_k$ determine the shape μ of (V, W)

$$\mu_1 = \mathrm{LIS}_1 \qquad \mu'_1 = \mathrm{LDS}_1$$

$$\mu_1 + \cdots + \mu_\ell = \mathrm{LIS}_\ell \qquad \mu'_1 + \cdots + \mu'_\ell = \mathrm{LDS}_\ell$$

Greene invariants



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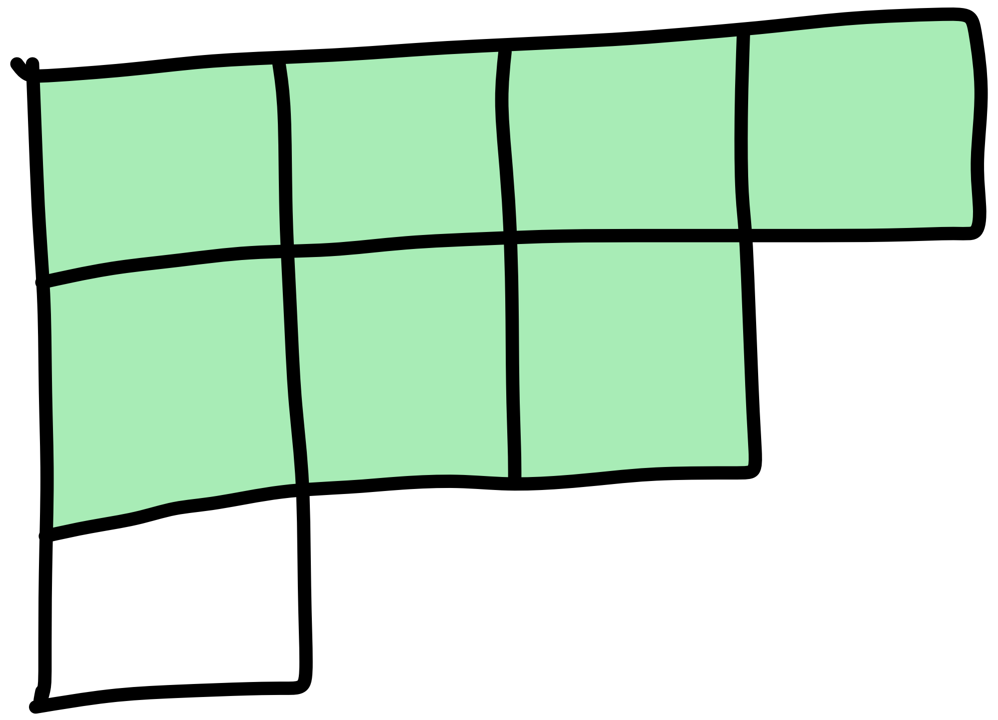

Theorem [IMS'21]

$\mathbf{LDS}_k$ and $\mathbf{LIS}_k$ determine the shape μ of (V, W)

$$\mu_1 = \mathrm{LIS}_1 \qquad \mu_1' = \mathrm{LDS}_1$$

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Greene invariants

- **Definition:**
  - $\mathbf{LDS}_k$ = maximal number of bullets in k nonintersecting loops
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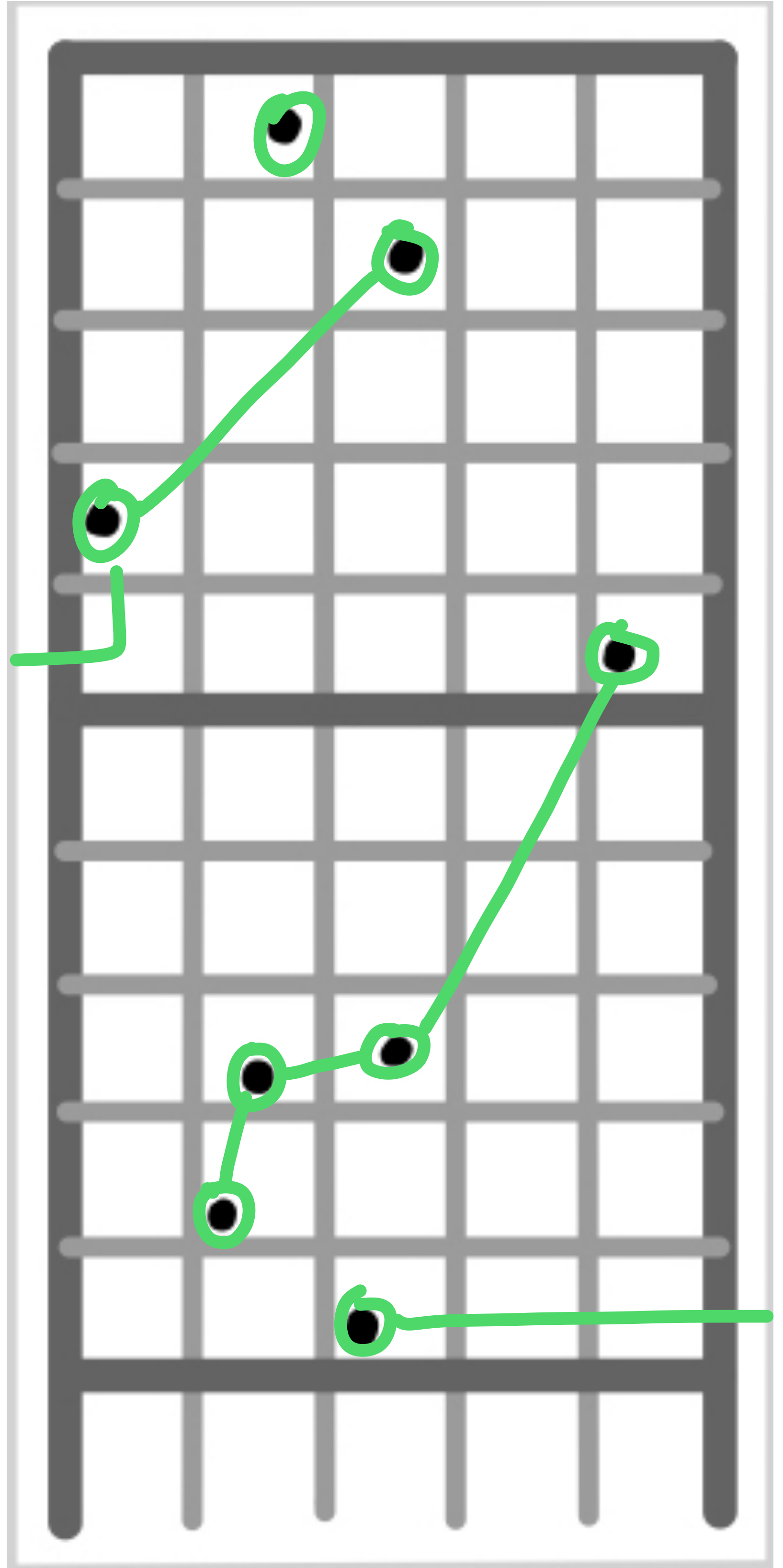

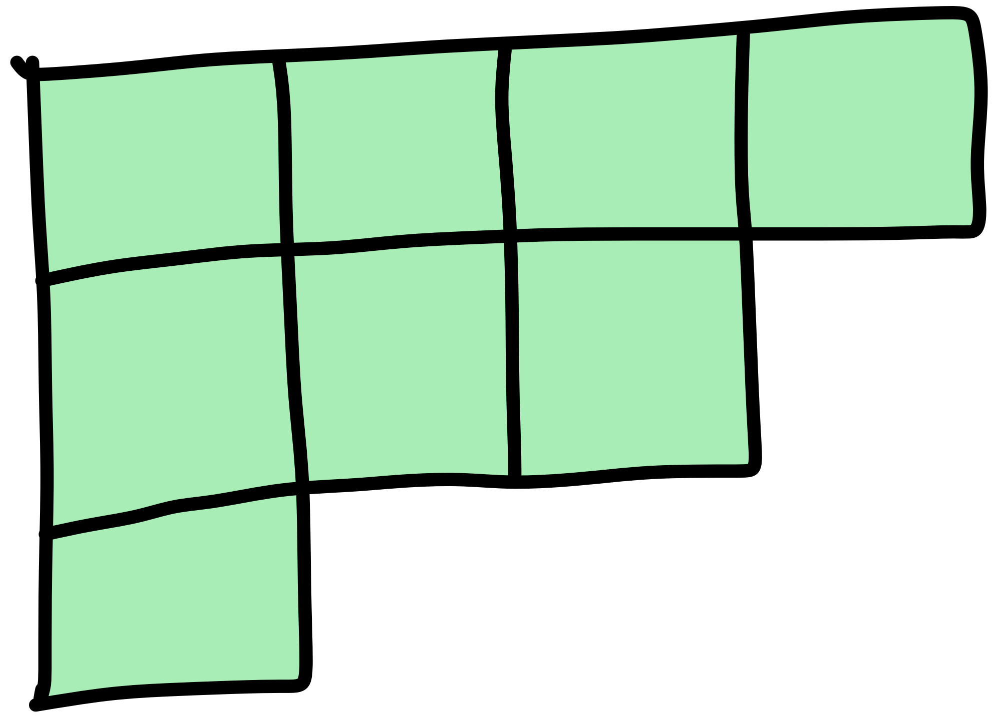

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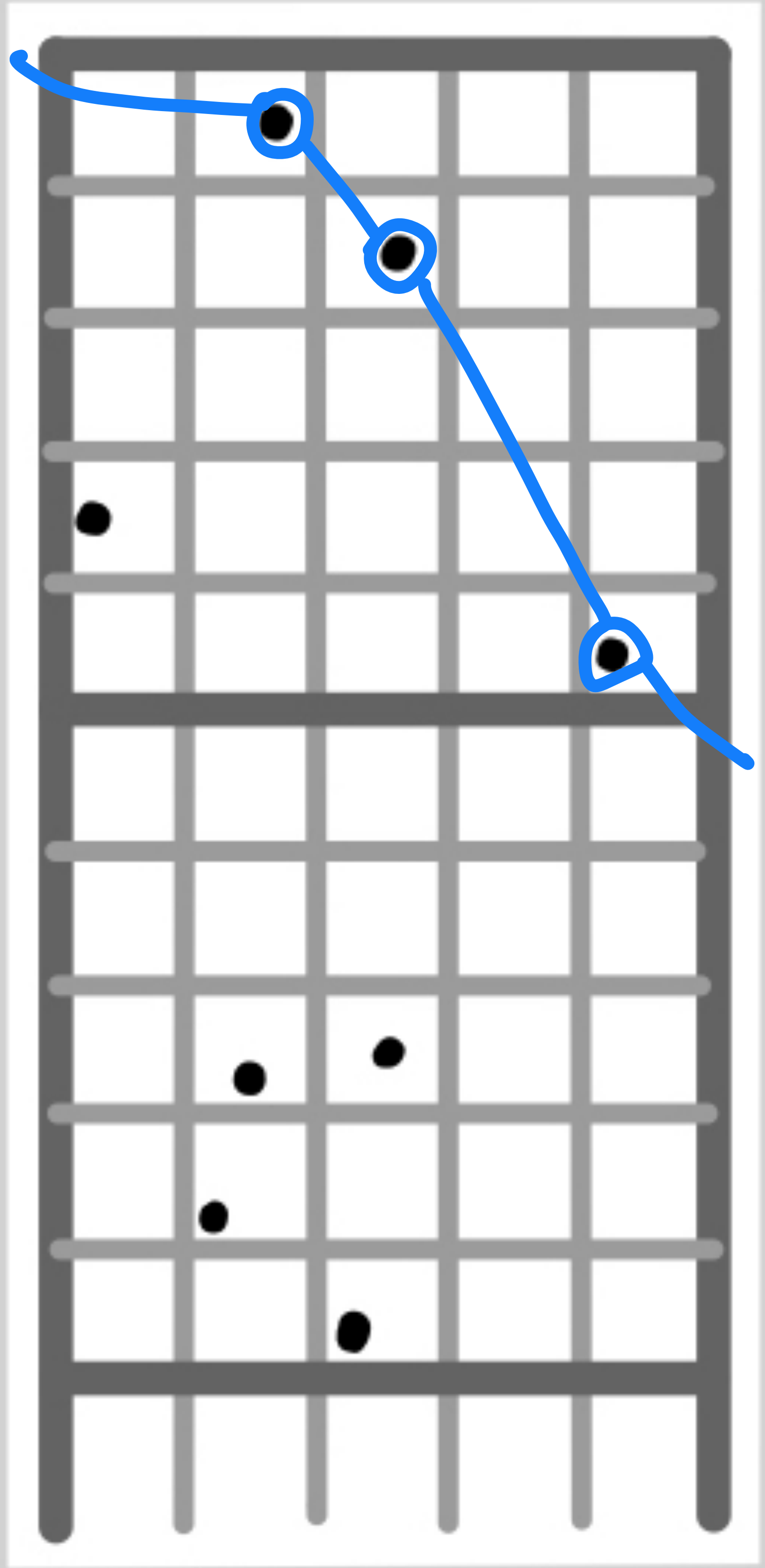

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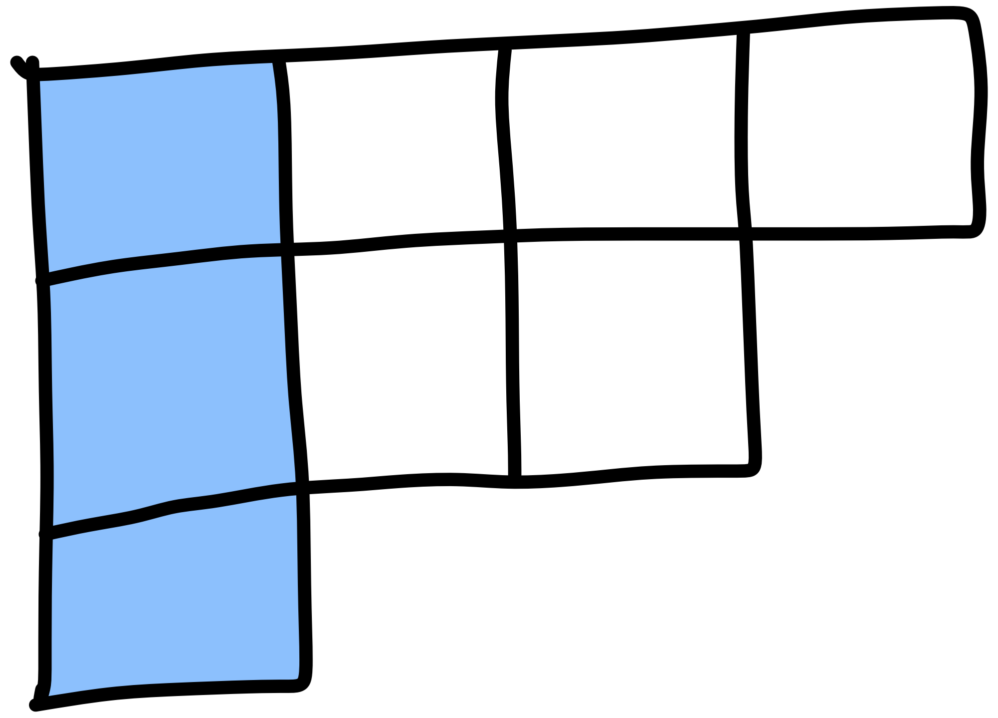

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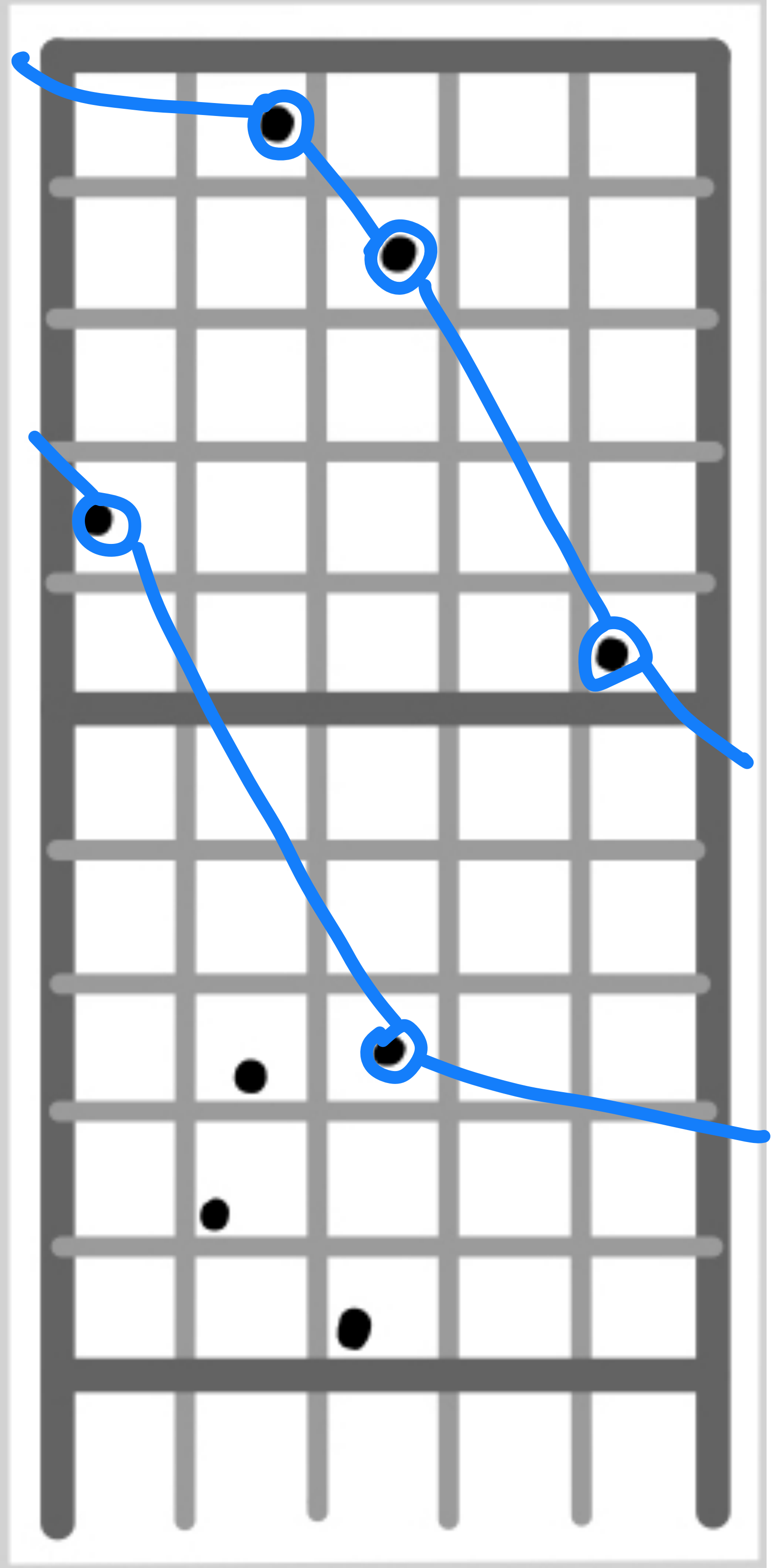

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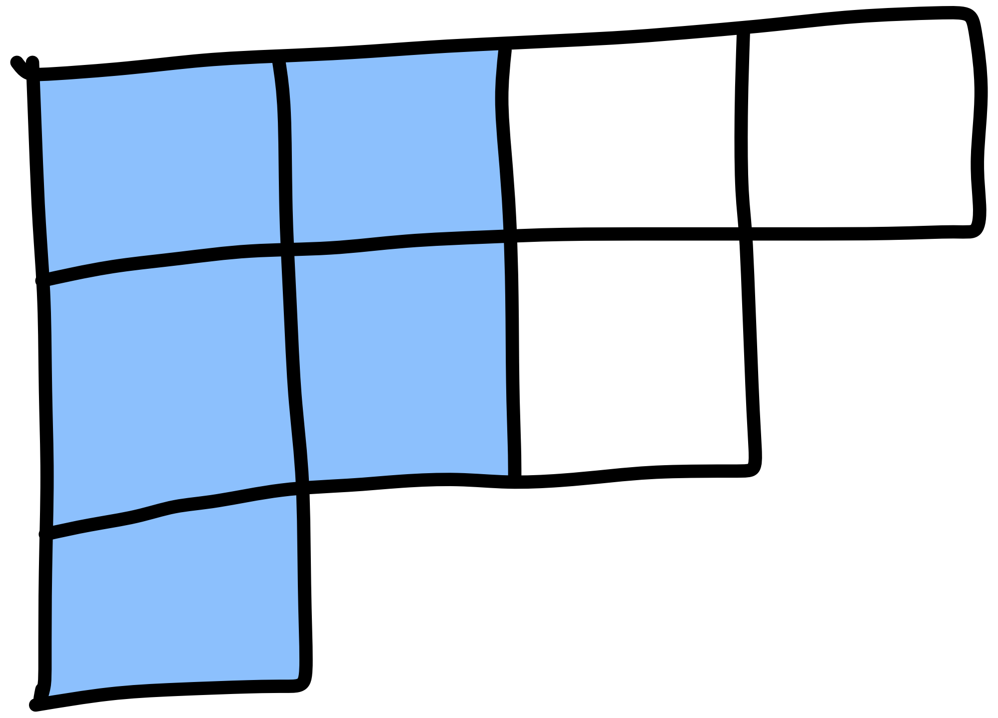

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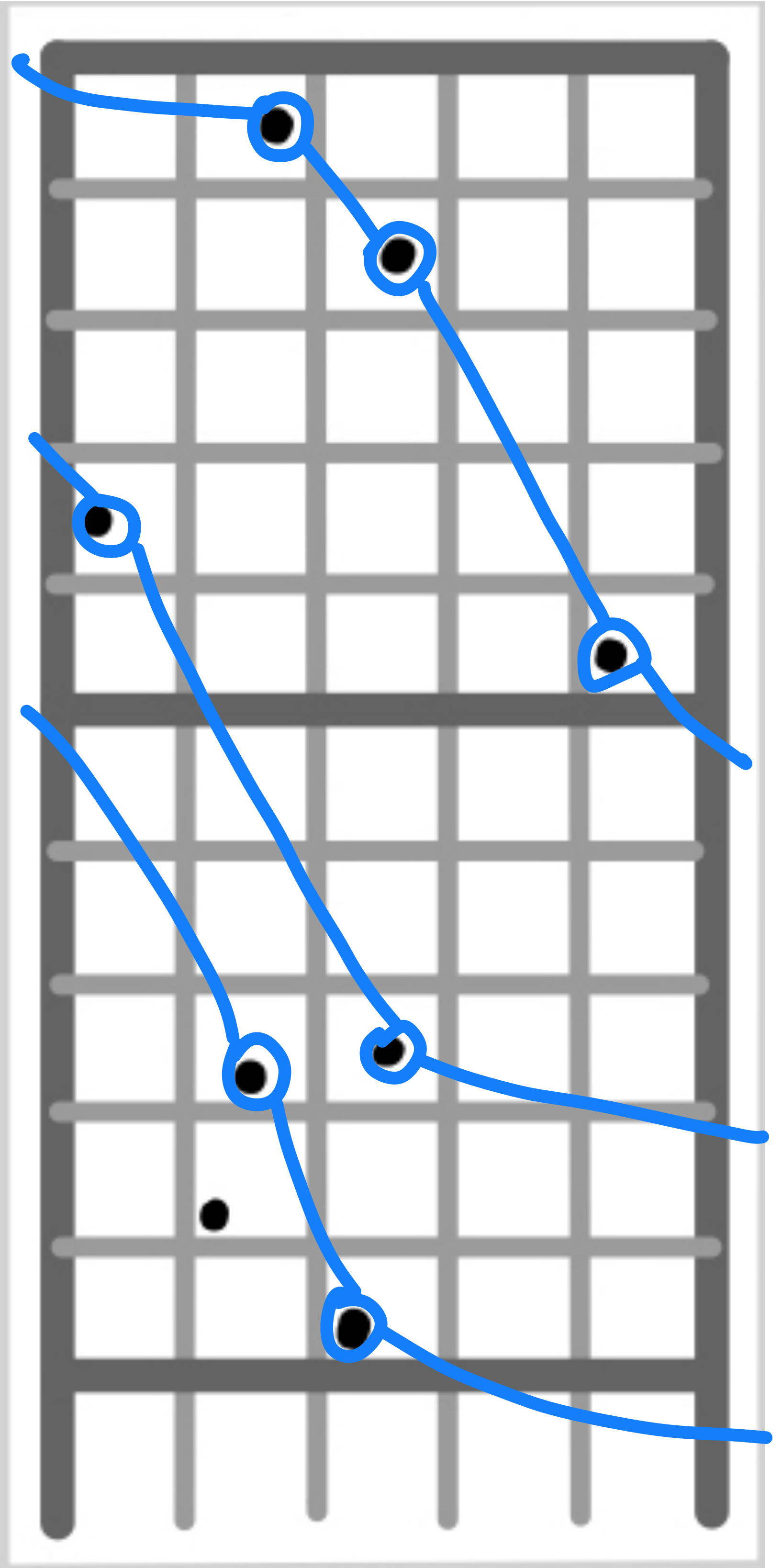

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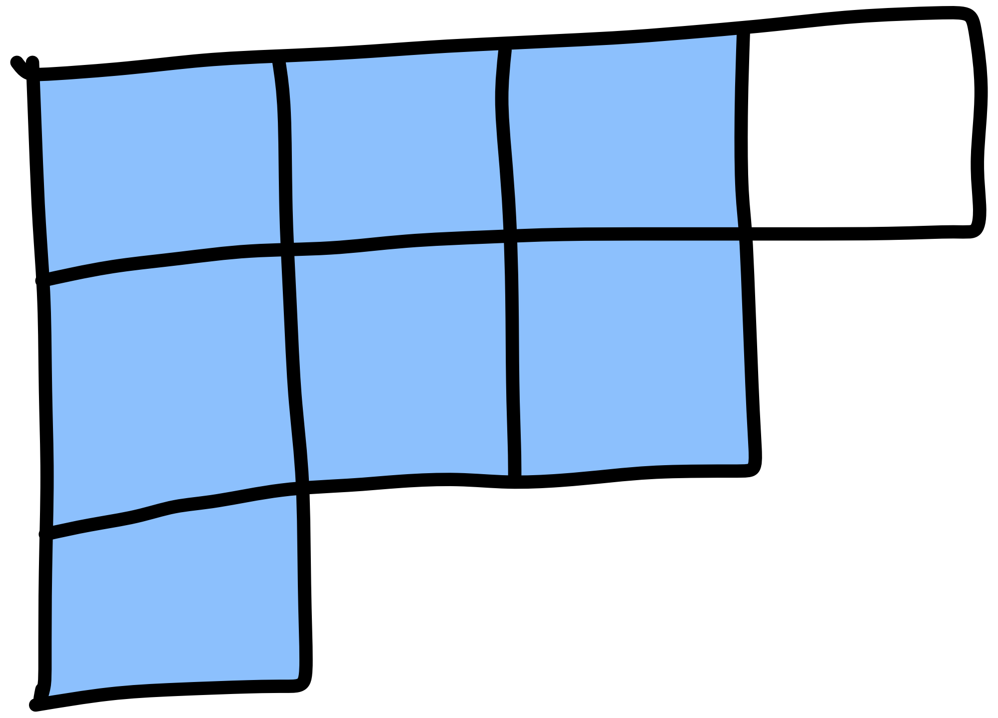

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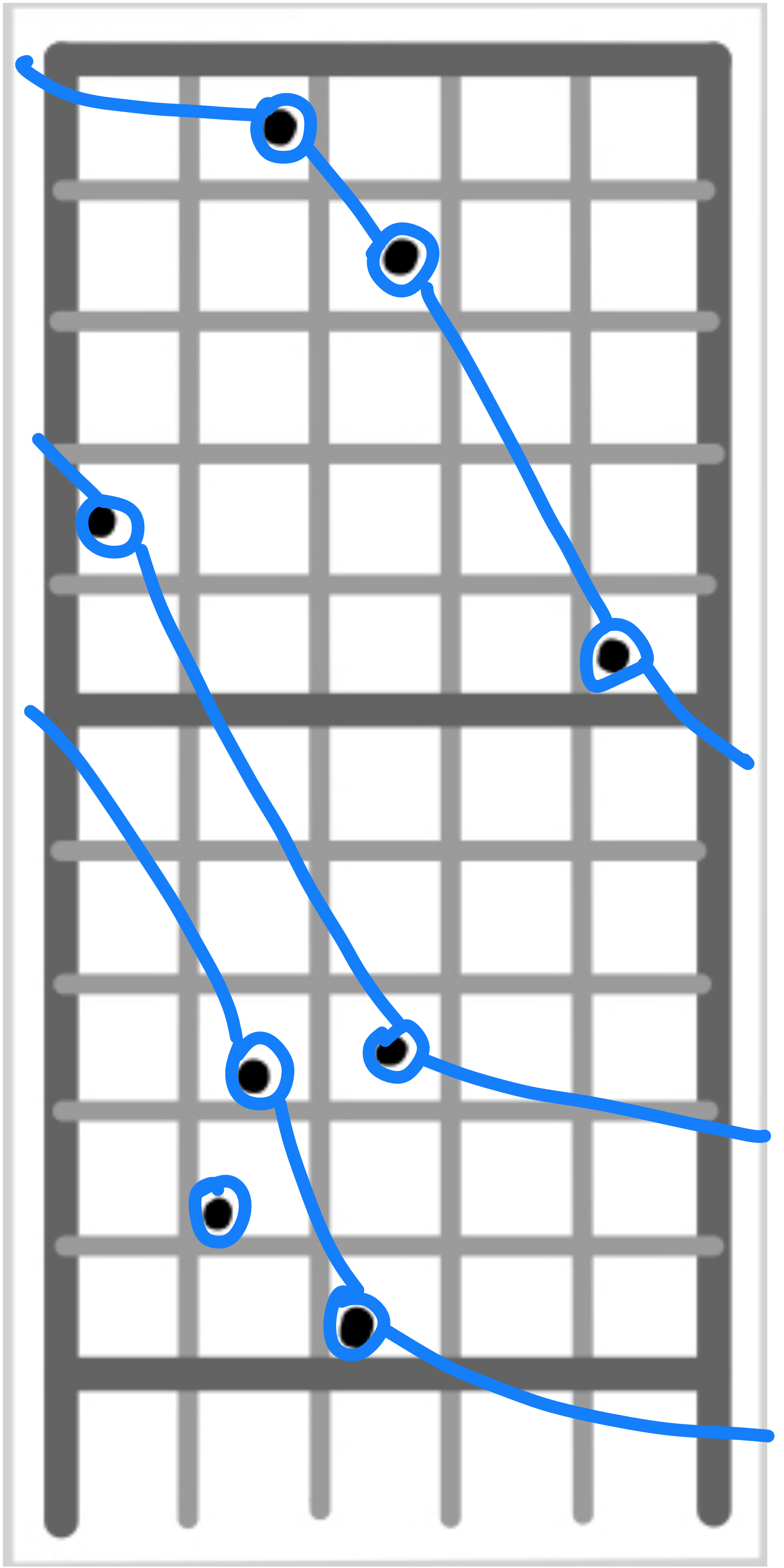

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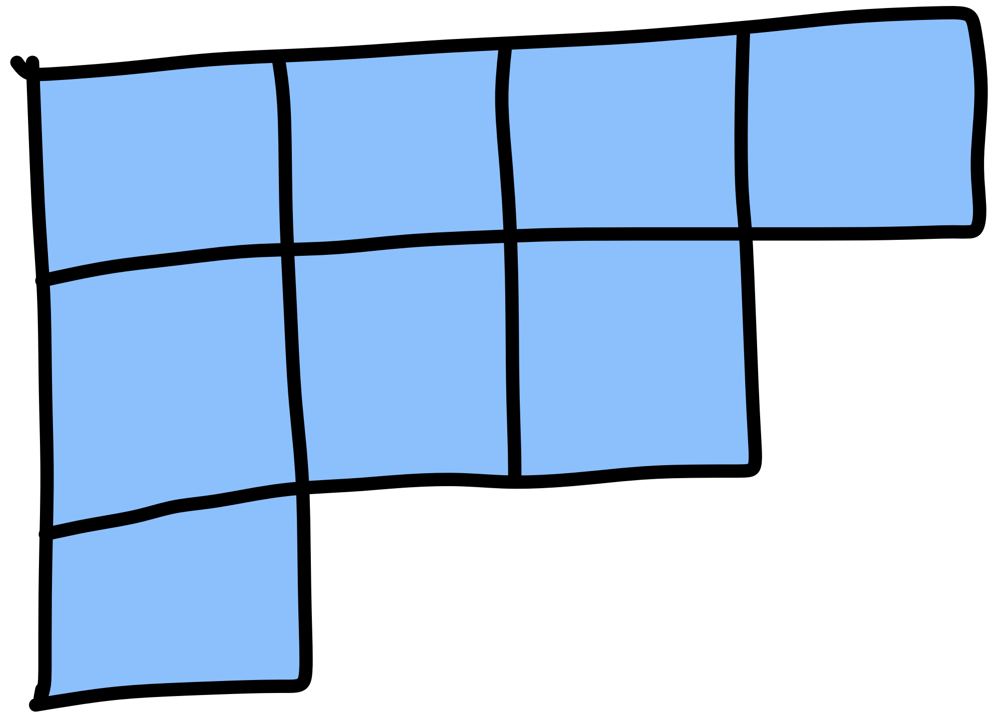

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Conclusion

- We construct a bijective q-extension of the RSK correspondence Υ
- With Υ we can prove bijectively the Cauchy identities (CI) for q-Whittaker polynomials
- It is the first time a bijective proof is given for CI of Macdonald polynomials outside of the Schur case
- Symmetries of the bijection Υ allow to prove Littlewood identities, Kawanaka identities and skew Gessel identities
- We extend in periodic setting the notion of Greene invariants