Positive temperature free fermions and solvable models in the KPZ class

Séminaire de physique mathématique
Institut de Physique Théorique, Paris
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Matteo Mucciconi — based on a collaboration with Takashi Imamura and Tomohiro Sasamoto

arXiv:2106.11922[math.CO]
arXiv:2106.11913[math.CO]

Plan of the talk

- **Part 1.** Presenting the problem we solve: connections between KPZ equation and determinantal/pfaffian point processes
- **Part 2.** Combinatorial construction of the correspondence KPZ models - free fermionic systems

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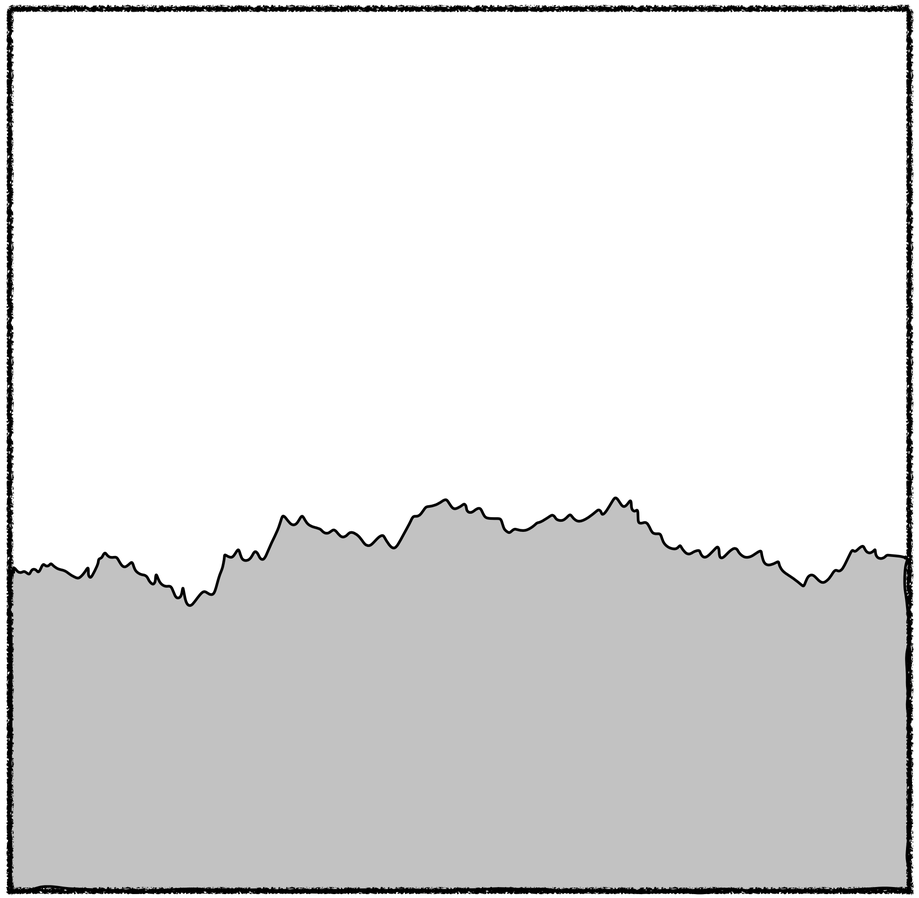
- **Part 1.** Presenting the problem we solve: connections between KPZ equation and determinantal/pfaffian point processes

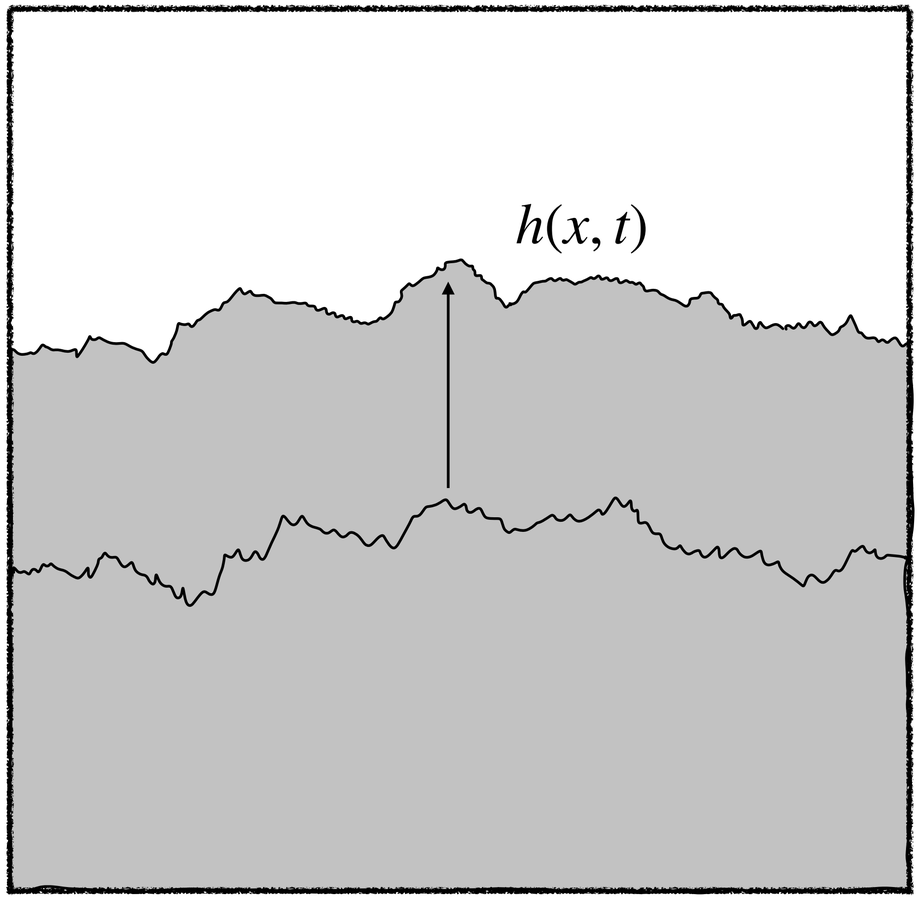
KPZ equation [Kardar-Parisi-Zhang '86]



$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}(\partial_x h)^2 + \eta$$

- h = random height function
  η = space-time white noise

KPZ equation [Kardar-Parisi-Zhang '86]



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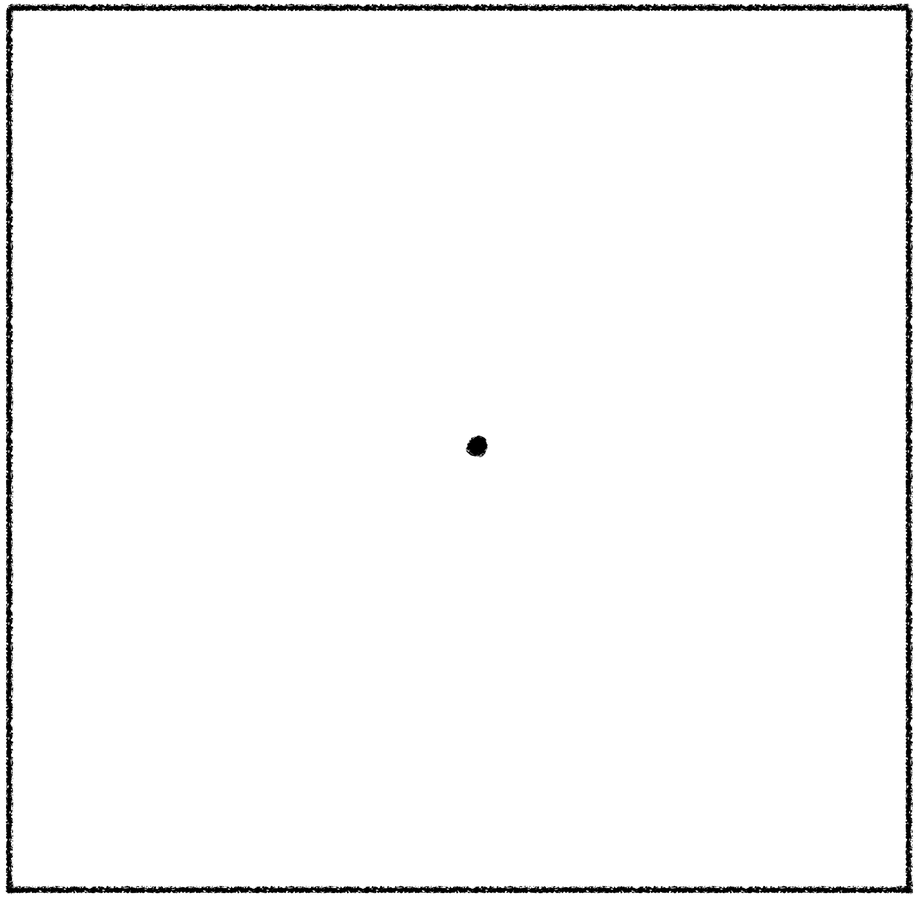
- h = random height function
  η = space-time white noise
- Well posedness: [Bertini-Giacomin '97], [Hairer '11], [Gubinelli-Perkowski-Imkeller '12]

KPZ equation : exact solutions



- Narrow wedge initial conditions

$$\begin{cases} \partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}(\partial_x h)^2 + \eta \\ h(x,0) = \log(\delta_x) \end{cases}$$

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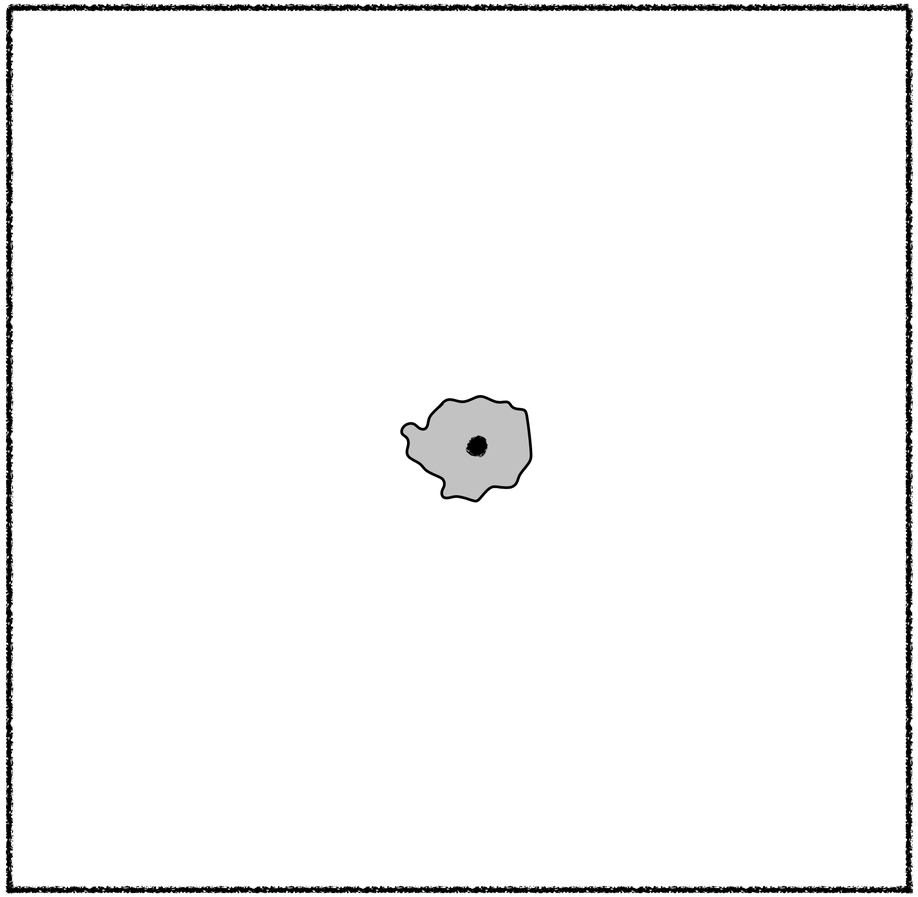

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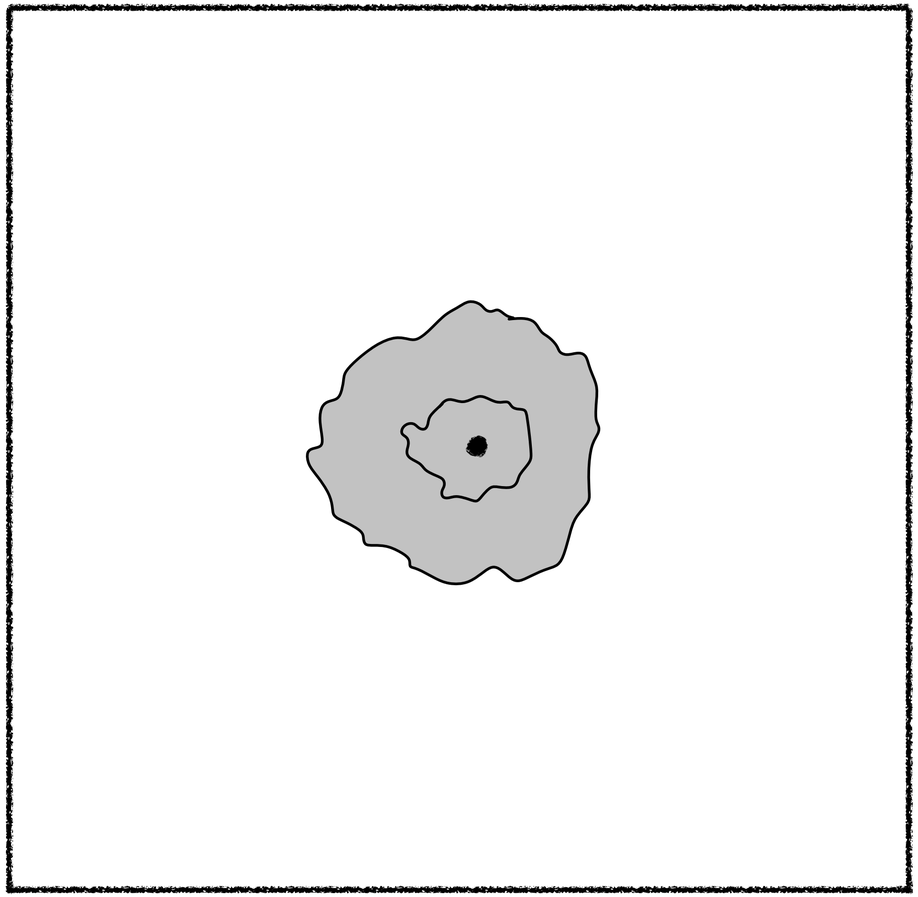

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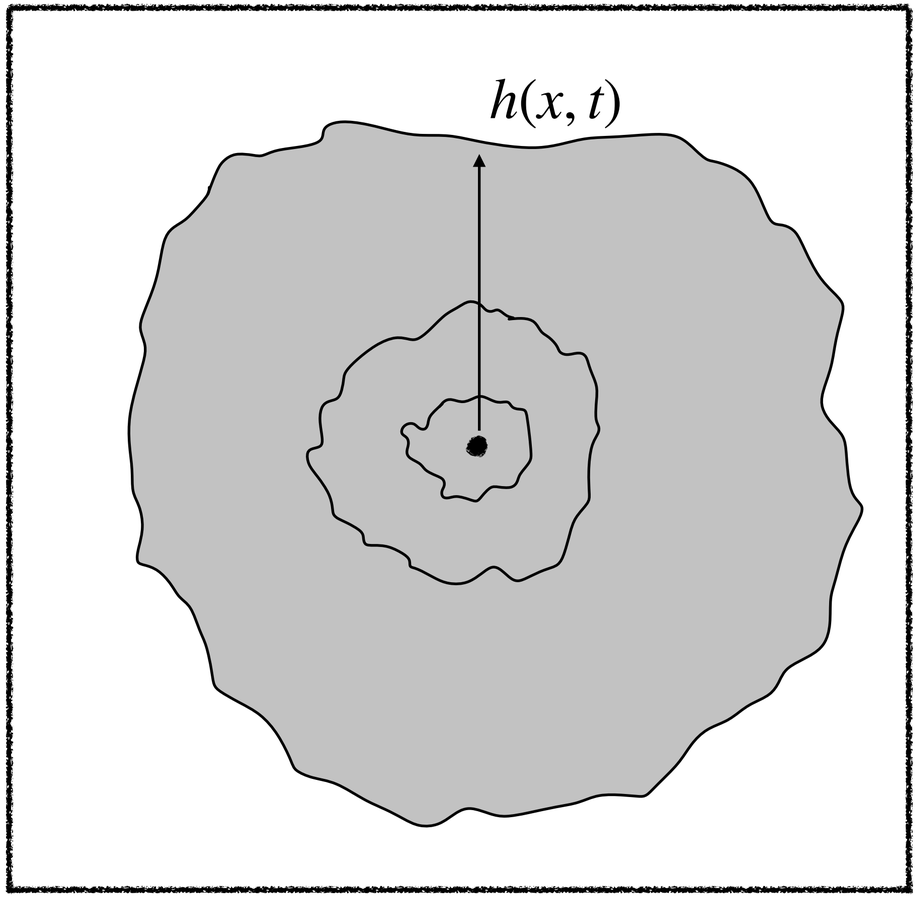

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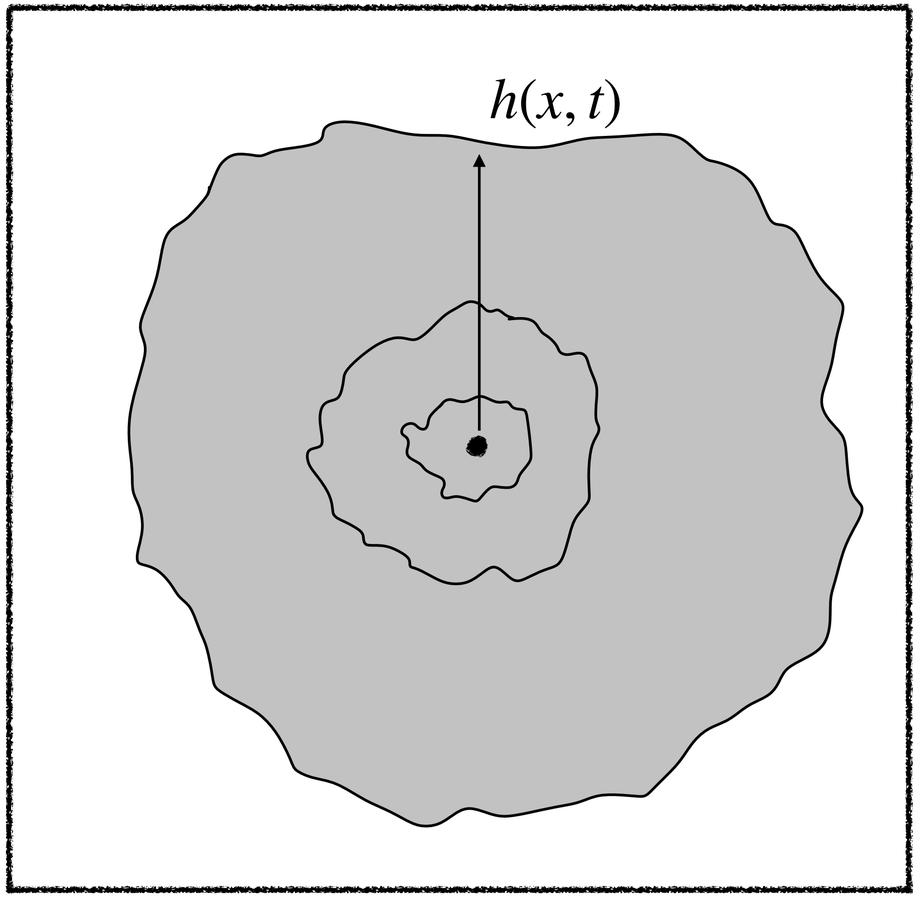

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- [Amir-Corwin-Quastel, Calabrese-Le Doussal, Dotsenko, Sasamoto-Spohn '11]

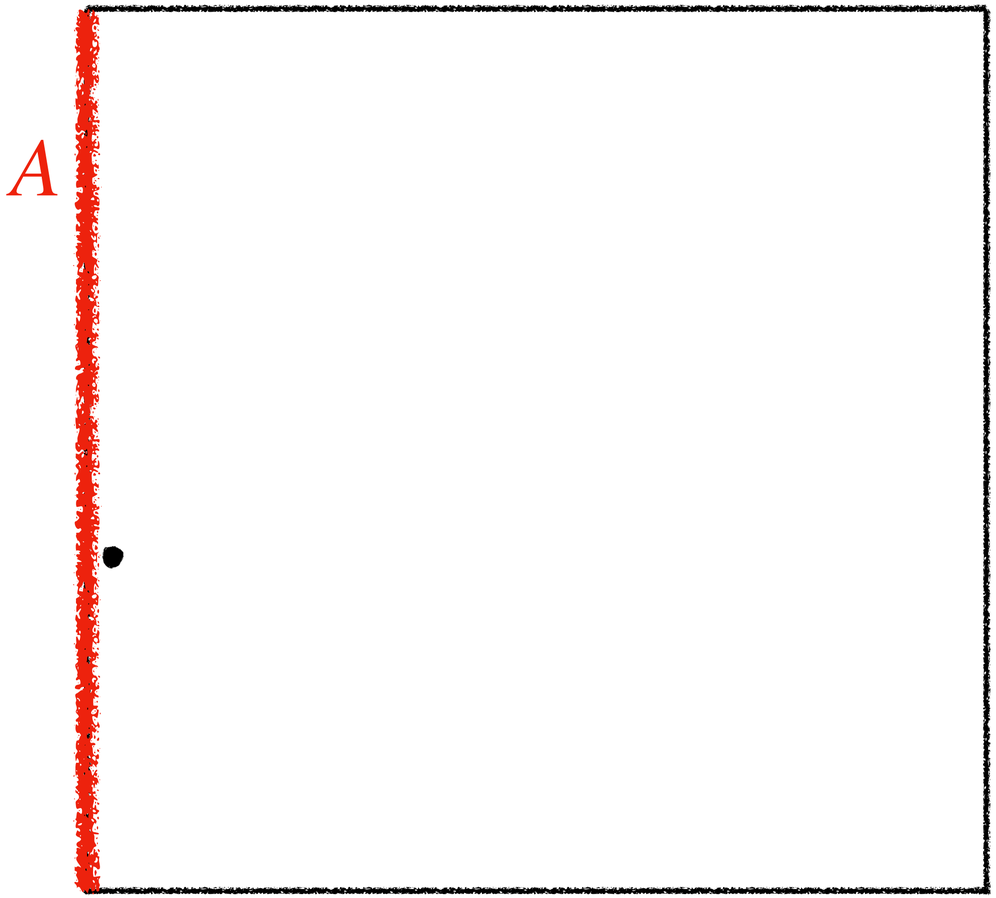
$$\mathbb{E}\left[\exp\left(-ze^{h(0,t)+t/24}\right)\right] = \det\left(1 - f K_{\text{Airy}}\right)_{\mathscr{L}^2(\mathbb{R})}$$

$$K_{\text{Airy}}(x,y) = \int_0^\infty \text{Ai}(x+z)\text{Ai}(y+z)dz \qquad f(x) = \frac{1}{1+e^{-xt^{1/3}}/z}$$

Airy Kernel

Fermi factor

KPZ equation : exact solutions



- Narrow wedge initial conditions in half space

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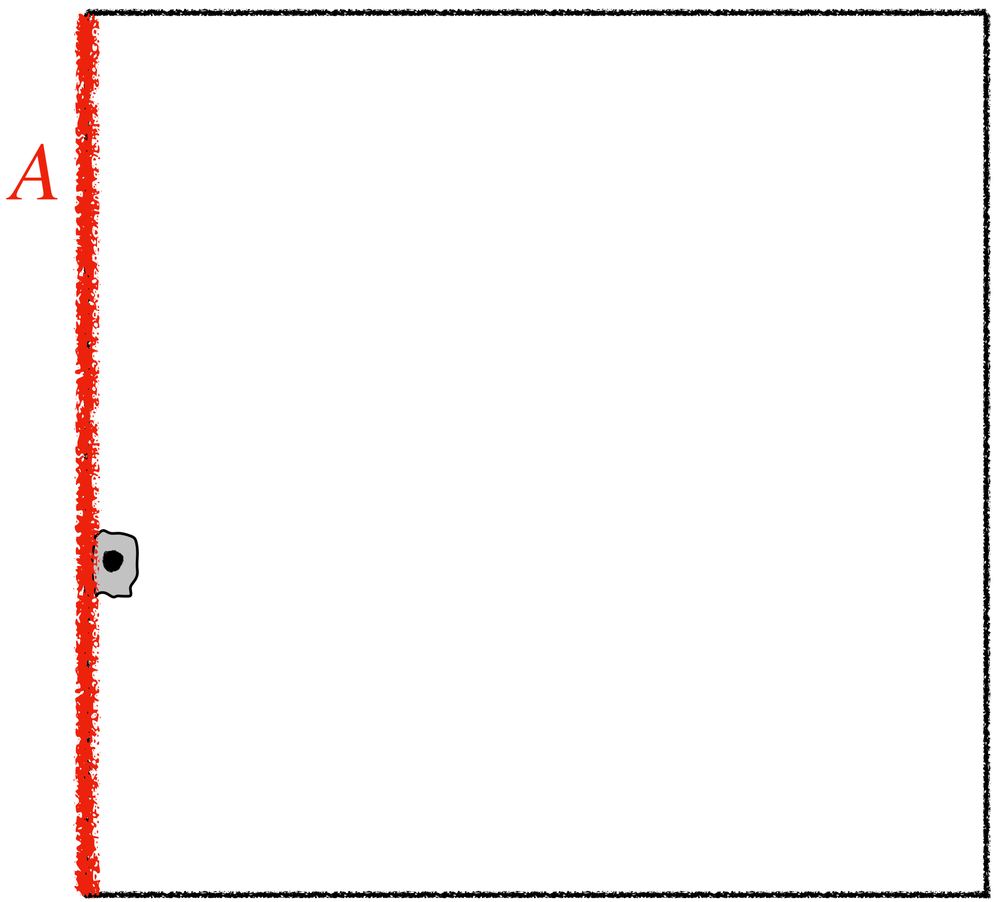

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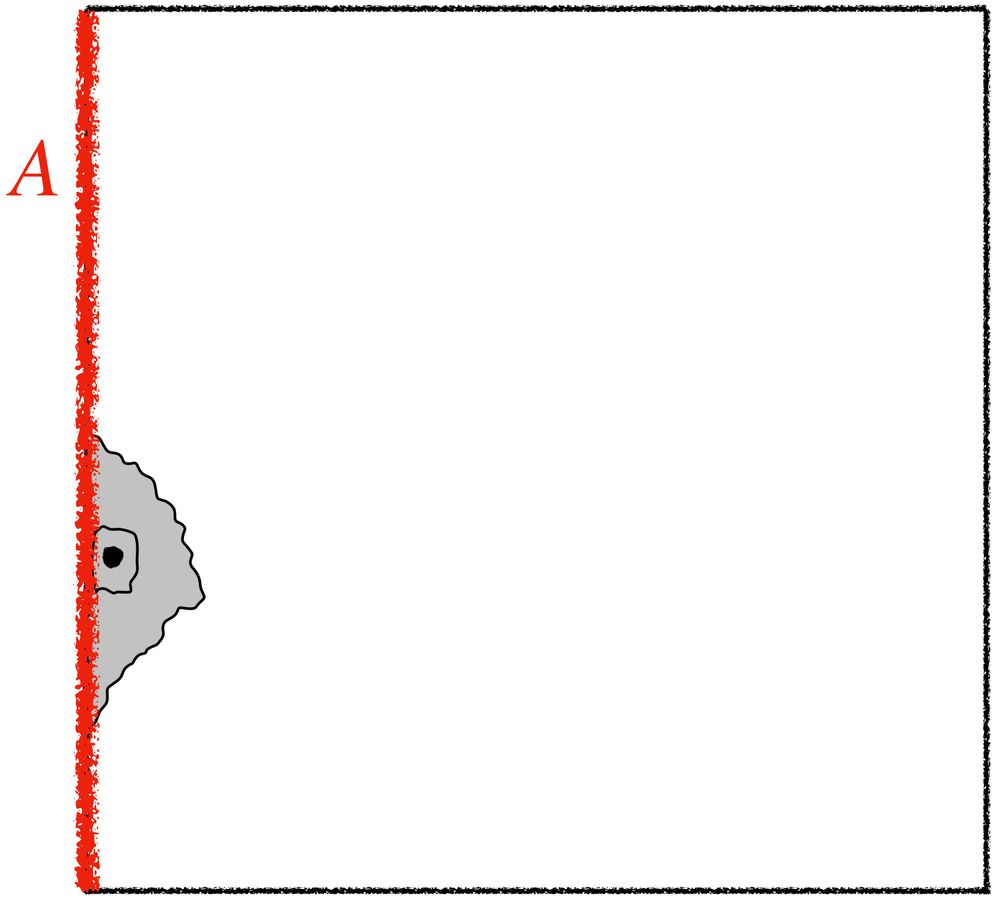

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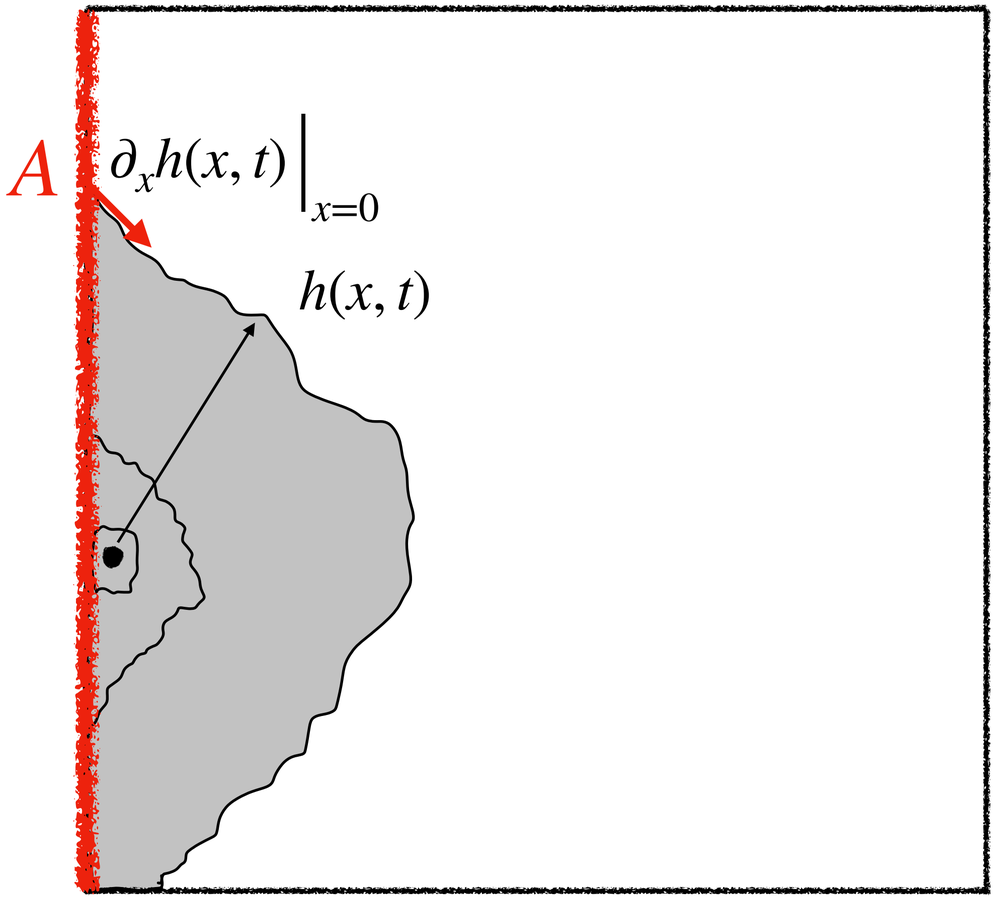

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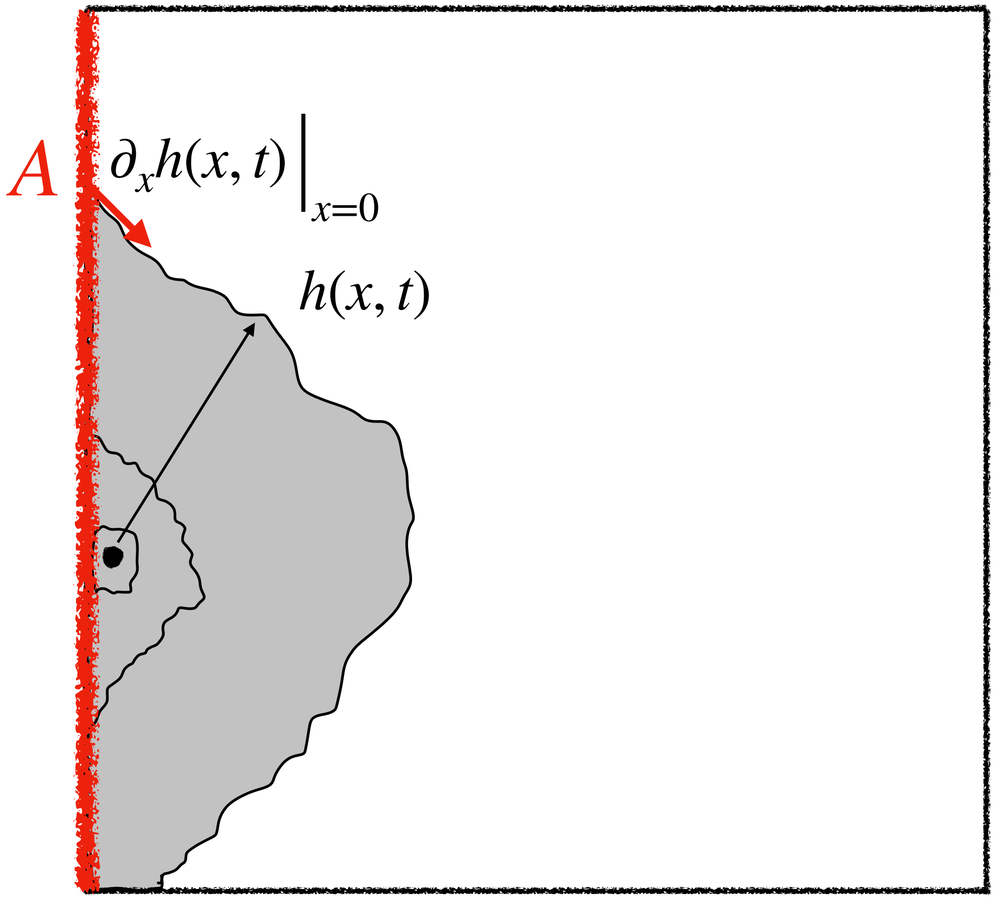

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- [Gueudre-Le Doussal'12, Borodin-Bufetov-Corwin'16, Barraquand-Borodin-Corwin-Wheeler'17 (*A=-1/2*), Krejenbrink-Le Doussal'19, De Nardis-Krejenbrink-Le Doussal-Thiery'20]

$$\mathbb{E}_{\text{hs}}\left[\exp\left(-ze^{h(0,t)+t/24}\right)\right] = \text{Pf}\left(1 - fK\right)_{\mathscr{L}^2(\mathbb{R})}$$

$K(x,y)$

2x2 matrix kernel
(Airy-like)

$$f(x) = \frac{1}{1 + e^{-xt^{1/3}}/z}$$

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KPZ equation : exact solutions

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- Apparent only from the solutions
- Can we establish connections between KPZ eq. and free fermions a priori?

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- Solutions are obtained through Bethe Ansatz (BA)
- BA is very powerful but requires difficult calculations and (often) non-rigorous arguments
- Can we create an elementary theory to solve the KPZ eq.?

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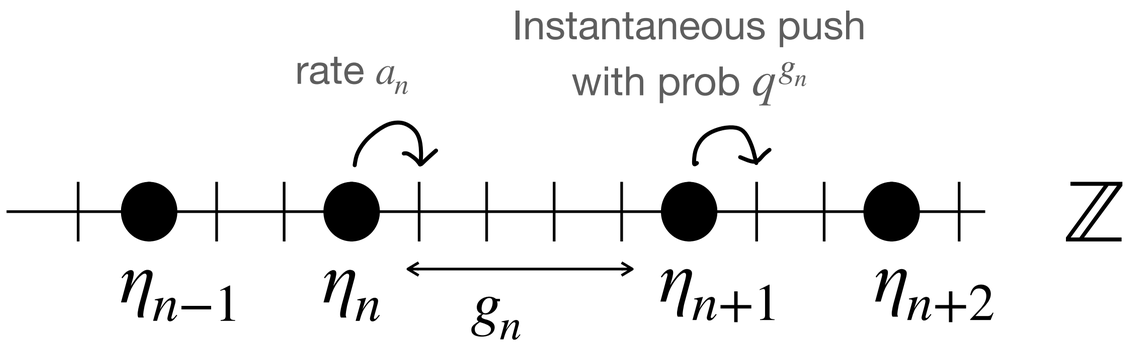
- …how to solve discrete models?

Solvable KPZ models and symmetric functions

- Typical model (FULL SPACE): *q-Push TASEP* [Borodin-Petrov '12]



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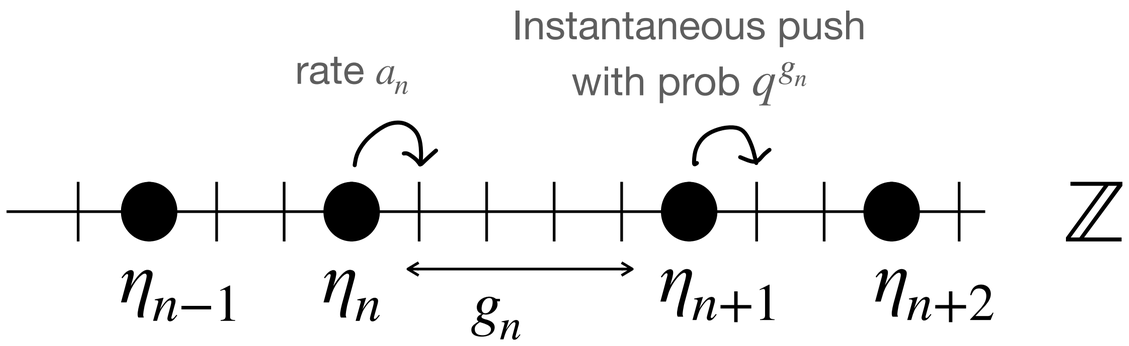

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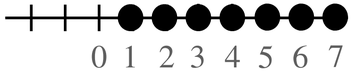

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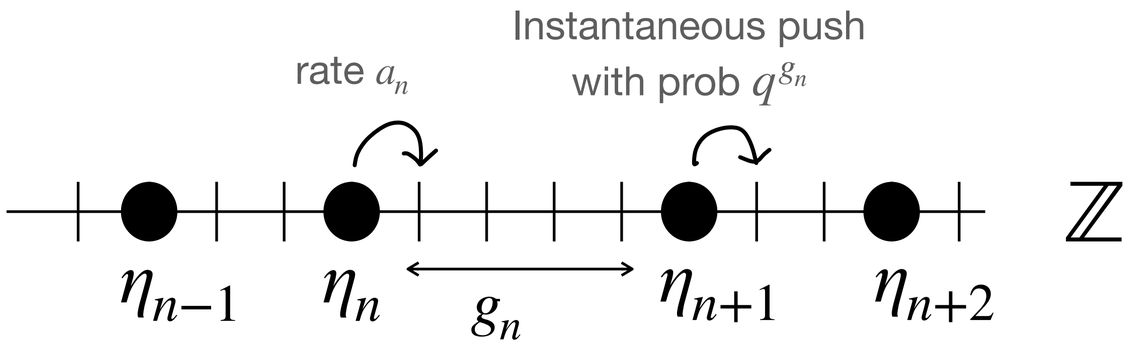

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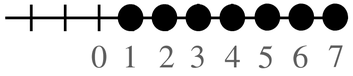

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q-Whittaker measure
[Borodin-Corwin '11]

$\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq 0)$

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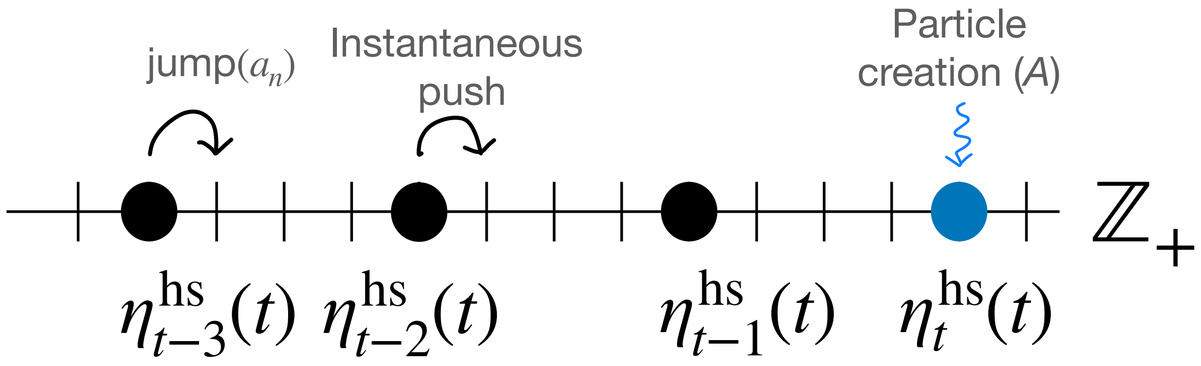

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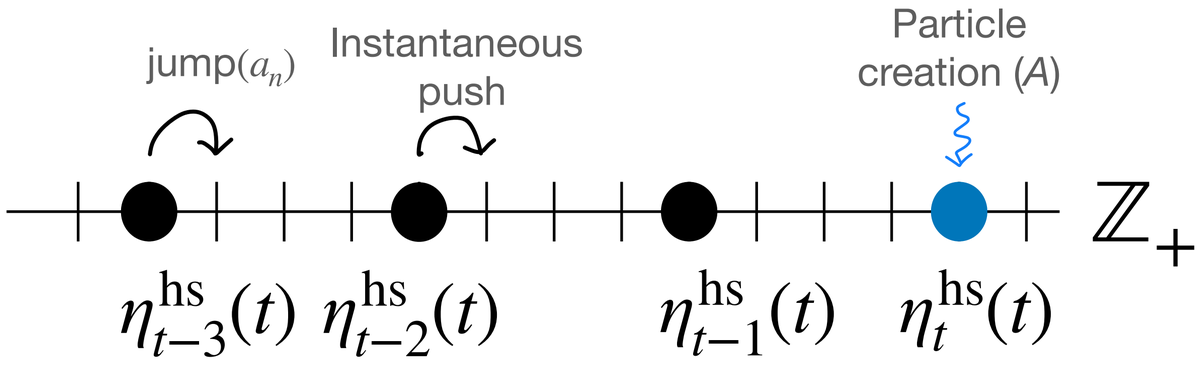

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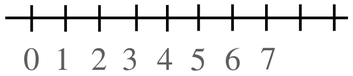

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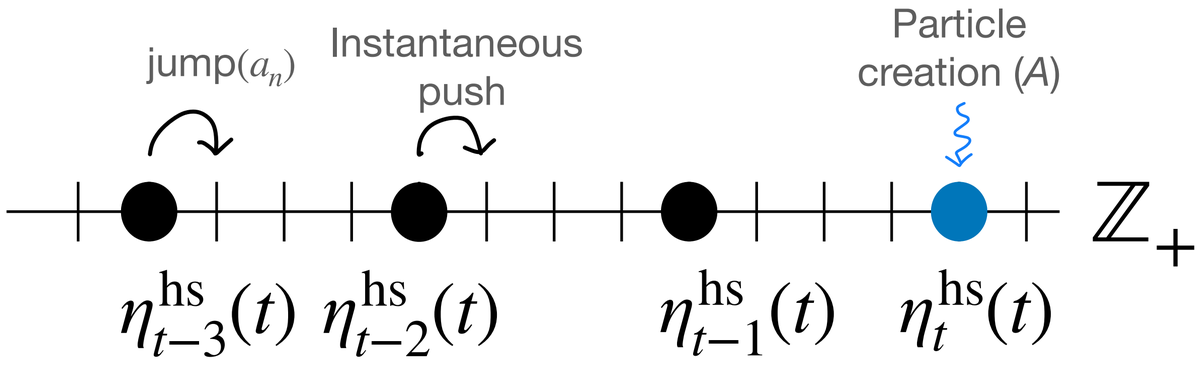

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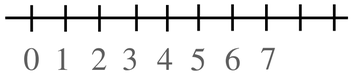

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Half space q-Whittaker measure [BBC '18]

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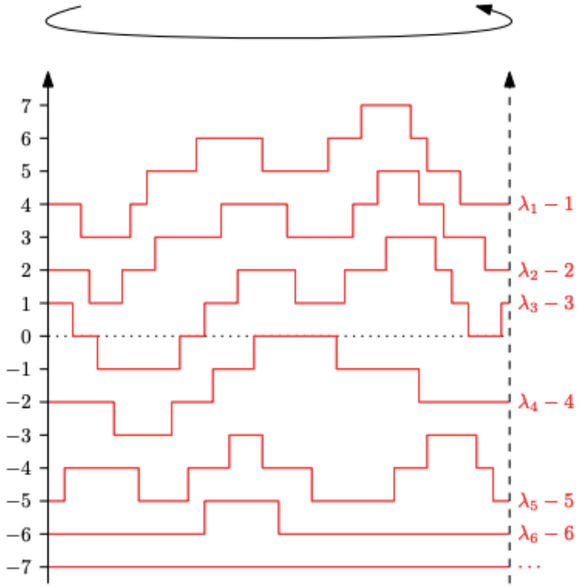
- **GOAL** : relate μ_1 with natural statistics of a determinantal/pfaffian point process

Positive temperature Free Fermions

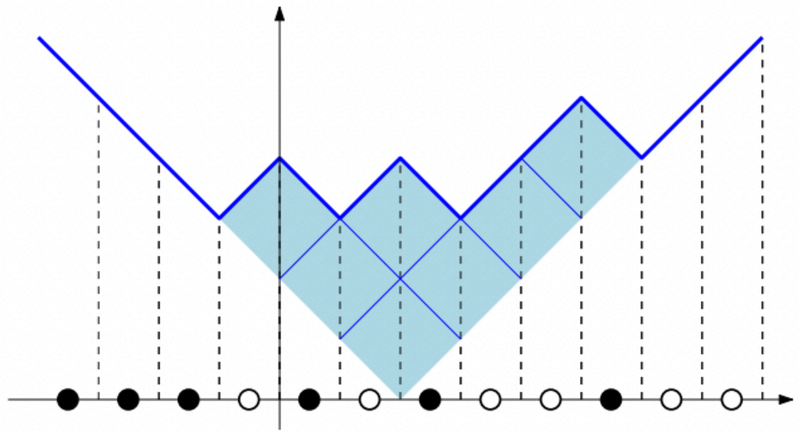
- Periodic Schur measure [Borodin '06]

$$\mathbb{P}(\lambda) = \frac{1}{\tilde{Z}^q_{a,b}} \sum_{\rho} q^{|\rho|} s_{\lambda/\rho}(a) s_{\lambda/\rho}(b)$$

$$s_{\lambda/\rho}(a) = \det \left[h_{\lambda_i - \rho_j - i + j}(a) \right]_{i,j}$$ Schur polynomials

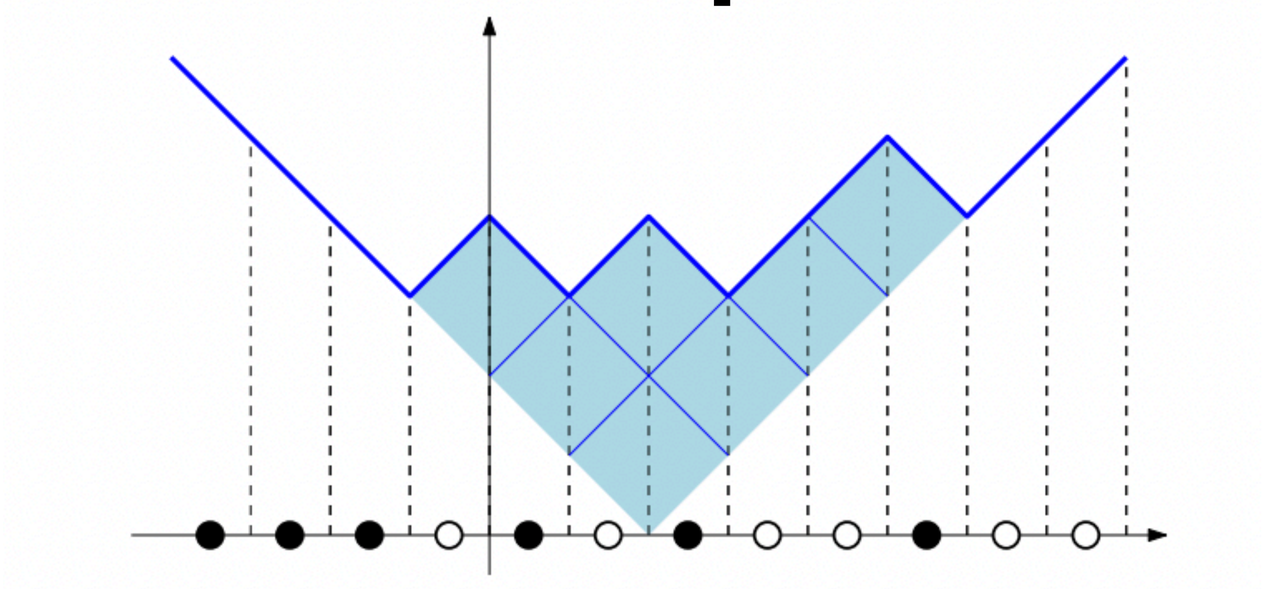


- $\mathbb{P}(S = k) \propto q^{k^2/2} t^k$ for $k \in \mathbb{Z}$ independent of λ
- $(\lambda_1 + S, \lambda_2 + S, \lambda_3 + S, \ldots)$ is a determinantal point process



Figures from [Betea-Bouttier]

Positive temperature Free Fermions



$$\mathbb{P}(\lambda_1 + S < s) = \det(1 - K)_{\ell^2\{s, s+1, \ldots\}}$$

- $(\lambda_1 + S,\ \lambda_2 + S,\ \lambda_3 + S, \ldots)$ is a determinantal point process with correlation kernel

$$K(x, y) = \frac{1}{(2\pi \mathrm{i})^2} \oint_{|z|=r} \frac{dz}{z^{x+1}} \oint_{|w|=r'} \frac{dw}{w^{-y+1}} \frac{F(z)}{F(w)} \kappa(z, w),$$

$$F(z) = \prod_{i \geq 1} \frac{(b_i/z; q)_\infty}{(a_i z; q)_\infty} \qquad \kappa(z, w) = \sqrt{\frac{w}{z}} \frac{(q; q)_\infty^2}{(z/w, qw/z; q)_\infty} \frac{\vartheta_3(\zeta z/w; q)}{\vartheta_3(\zeta; q)} \qquad (z; q)_\infty = \prod_{\ell \geq 0} (1 - q^\ell z)$$

Free boundary Schur measure

- Free boundary Schur measure [Betea-Bouttier-Nejjar-Vuletic '17]

$$\mathbb{P}(\lambda) = \frac{\mathbf{1}_{\lambda' \text{even}}}{\tilde{Z}^q_{a;A}} \sum_{\rho' \text{even}} q^{|\rho|/2} s_{\lambda/\rho}(a;A)$$

- $\mathbb{P}(S = k) \propto q^{2k^2} t^{2k}$ for $k \in \mathbb{Z}$ independent of λ

- $(\lambda_1 + 2S,\ \lambda_2 + 2S,\ \lambda_3 + 2S, \ldots)$ is a determinantal point process

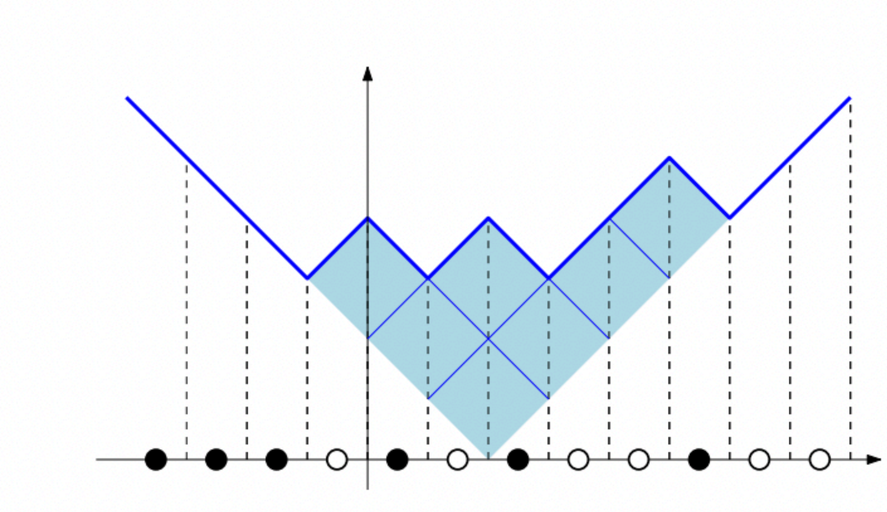
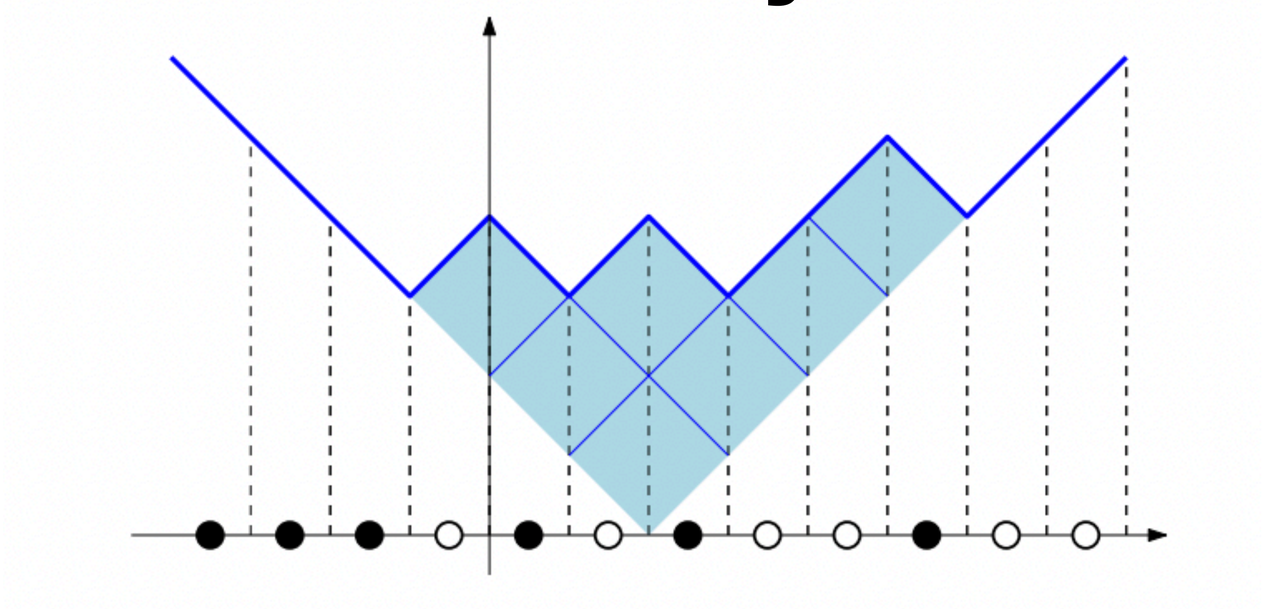


Figure from [Betea-Bouttier]

Free boundary Schur measure



$$\mathbb{P}(\lambda_1 + S < s) = \mathrm{Pf}(1 - L)_{\ell^2\{s,s+1,\ldots\}}$$

$$L(x,y) = \begin{pmatrix} k(x,y) & -\nabla_y k(x,y) \\ -\nabla_x k(x,y) & \nabla_x \nabla_y k(x,y) \end{pmatrix}$$

$$k(x,y) = \frac{1}{(2\pi \mathrm{i})^2} \oint_{|z|=r} \frac{dz}{z^{x+3/2}} \oint_{|w|=r} \frac{dw}{w^{y+5/2}} F(z)F(w)\kappa^{\mathrm{hs}}(z,w)$$

$$F(z) = \frac{(A/z;q)_\infty}{(Az;q)_\infty} \prod_{i\geq 1} \frac{(a_i/z;q)_\infty}{(a_i z;q)_\infty}$$

$$\kappa^{\mathrm{hs}}(z,w) = \frac{(q,q,w/z,qz/w;q)_\infty}{(1/z^2,1/w^2,1/zw,qwz;q)_\infty} \frac{\vartheta_3(\zeta^2 z^2 w^2;q^2)}{\vartheta_3(\zeta^2;q^2)}$$

	KPZ solvable models	Determinantal/Pfaffian point processes
Full space	$\mu \sim \dfrac{b_\mu \mathscr{P}_\mu(x)\mathscr{P}_\mu(y)}{Z^q_{x,y}}$	$\lambda \sim \dfrac{1}{\tilde{Z}^q_{x,y}} \sum_\rho q^{\lvert\rho\rvert} s_{\lambda/\rho}(x) s_{\lambda/\rho}(y)$
Half space	$\mu^{\mathrm{hs}} \sim \dfrac{b^{\mathrm{el}}_\mu \mathscr{P}_\mu(a;A)}{Z^q_{a;A}}$	$\lambda^{\mathrm{hs}} \sim \dfrac{\mathbf{1}_{\lambda'\mathrm{even}}}{\tilde{Z}^q_{a;A}} \sum_{\rho'\mathrm{even}} q^{\lvert\rho\rvert/2} s_{\lambda/\rho}(a;A)$

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THEOREM (Imamura-M.-Sasamoto '21)

$$\mu_1 + \chi \overset{\mathscr{D}}{=} \lambda_1 \qquad\qquad \mu_1^{\mathrm{hs}} + \chi \overset{\mathscr{D}}{=} \lambda_1^{\mathrm{hs}} \qquad (\star)$$

χ independent of μ_1 and $\mathbb{P}(\chi = n) = q^n (q^{n+1}; q)_\infty$

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Comments on ($\star$):

- Reveals the origin of determinantal formulas for KPZ solvable models at positive temperature
- Nice symmetrical relation between full and half space
- Suggests a new paradigm to solve models
- Reveals combinatorial properties between Schur polynomials and q-Whittaker polynomials
- Earlier results relating KPZ models and free fermions:
    - [Dean-Le Doussal-Majumdar-Schehr'15]
    - [Borodin'16],[Borodin-Gorin'16],[Borodin-Ohlshanki'16],[Borodin-Corwin-Barraquand-Wheeler'17]

Plan of the talk

- **Part 2.** combinatorial construction of the correspondence KPZ models - free fermionic systems

- We prove ($\star$) combinatorially, developing a (bijective!) q-extension of the RSK.

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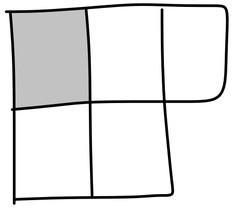
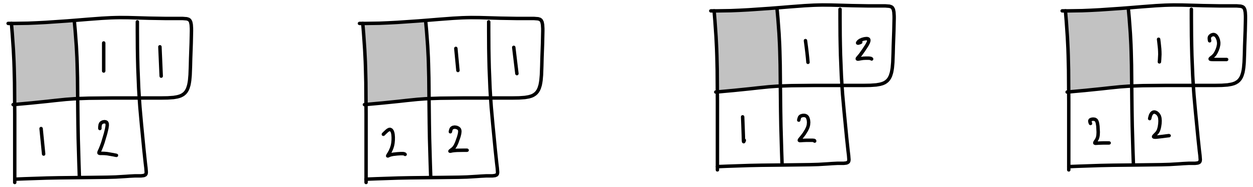
$\lambda/\rho =$ 



$$s_{\lambda/\rho}(x_1, x_2) = x_1^3 x_2 + x_1^2 x_2^2 + x_1^2 x_2^2 + x_1 x_2^3$$

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- **Combinatorial formulas**

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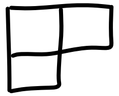
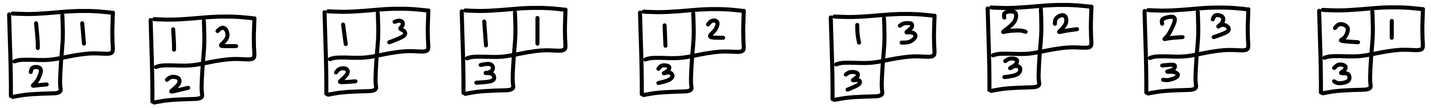
  - $$\mathscr{P}_\mu(x;q) = \sum_{V \in VST(\mu)} q^{\mathscr{H}(V)} x^V$$ sum over "vertically strict tableaux"

    $\mathscr{H}$ = intrinsic energy

  Example : $\mu =$  $n = 3$

$V =$ 

$\mathscr{H} =$ 0 0 0 0 0 0 0 0 1

$$\mathscr{P}_\mu(x;q) = x_1^2x_2 + x_1x_2^2 + x_1x_2x_3 + x_1^2x_3 + x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + qx_1x_2x_3$$

Cauchy Identities

- $$\sum_{\lambda,\rho} q^{|\rho|} s_{\lambda/\rho}(x) s_{\lambda/\rho}(y) = \frac{1}{(q;q)_\infty} \prod_{i,j} \frac{1}{(x_i y_j; q)_\infty}$$
- $$\sum_{\mu} b_\mu \mathscr{P}_\mu(x) \mathscr{P}_\mu(y) = \prod_{i,j} \frac{1}{(x_i y_j; q)_\infty}$$

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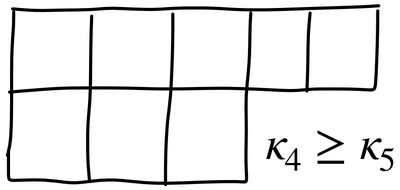
- $\mathscr{K}(\mu) = \{\kappa = (\kappa_1, \ldots, \kappa_{\mu_1}) : \kappa_i \geq \kappa_{i+1} \text{ if } \mu_i' = \mu_{i+1}'\}$

$$b_\mu = \prod_{i \geq 1} (q;q)^{-1}_{\mu_i - \mu_{i+1}} = \sum_{\kappa \in \mathscr{K}(\mu)} q^{\kappa_1 + \kappa_2 + \cdots + \kappa_{\mu_1}}$$

$\mu =$ 

$\kappa_4 \geq \kappa_5$

$\kappa_1 \geq \kappa_2 \geq \kappa_3$

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IDEA: $(P, Q) \xleftrightarrow{\Upsilon} (V, W; \kappa, \nu)$

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Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)

P

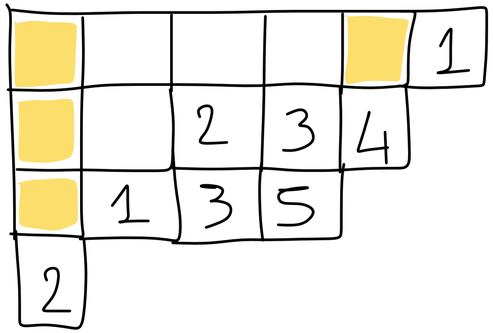
					1
		2	3	4	
	1	3	5		
2					

Q

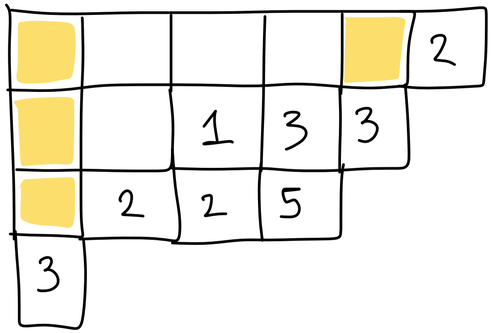
					2
		1	3	3	
	2	2	5		
3					

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)

P



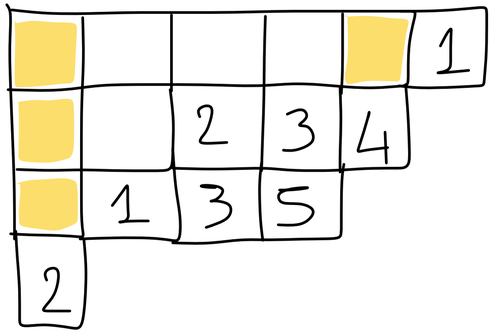
Q



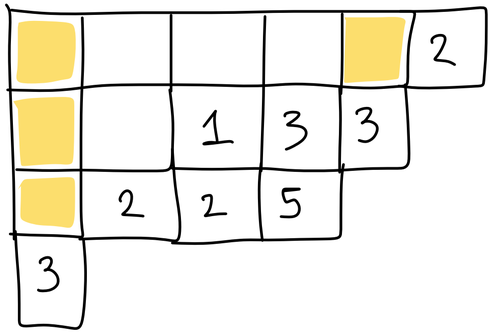
- First construct ν by “squeezing” P, Q

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)

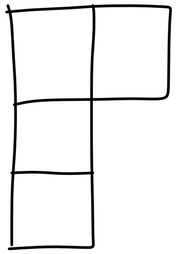
P



Q



$\nu =$



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Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)

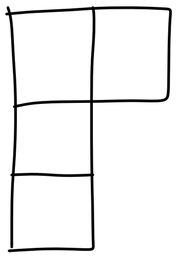
P

			1
	2	3	4
1	3	5	
2			

Q

			2
	1	3	3
2	2	5	
3			

$\nu =$



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Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)

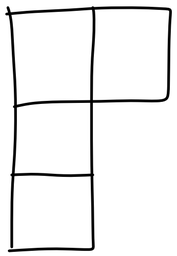
P

			1
	2	3	4
1	3	5	
2			

Q

			2
	1	3	3
2	2	5	
3			

$\nu =$



- First construct ν by “squeezing” P, Q
- From now on assume pair (P, Q) is “squeezed”

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)

P

			1
	2	3	4
1	3	5	
2			

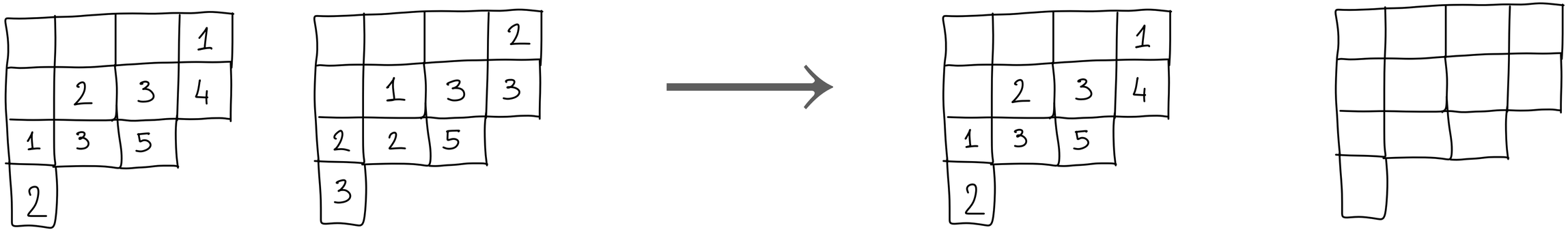
Q

			2
	1	3	3
2	2	5	
3			

- To construct (V, W) we define a dynamics on tableaux based on internal insertion [Sagan-Stanley'90]

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)

P $\qquad$ Q



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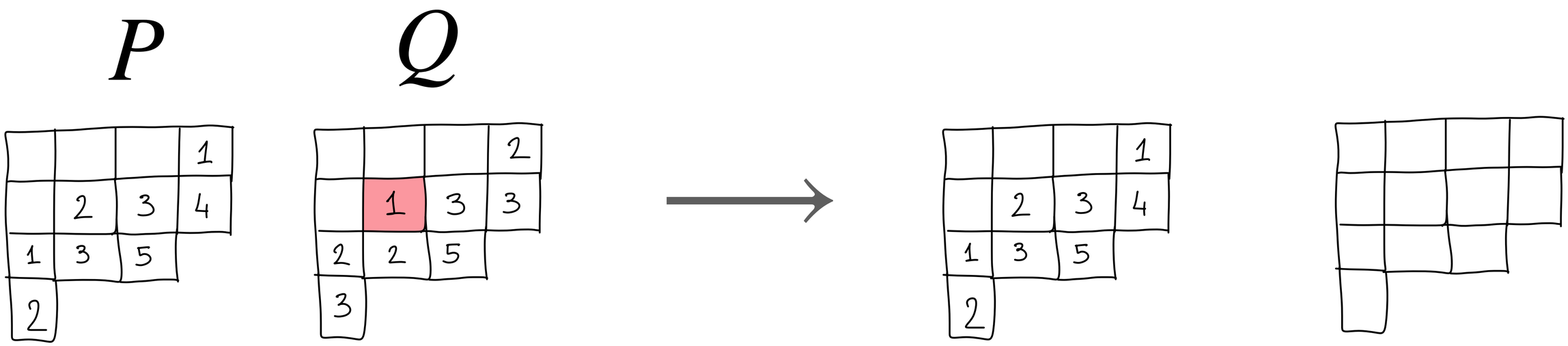

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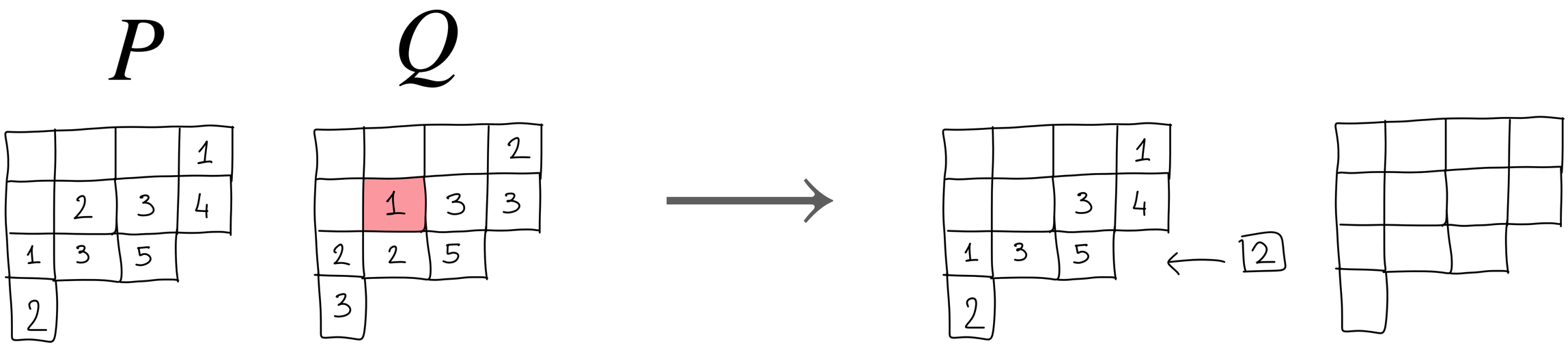

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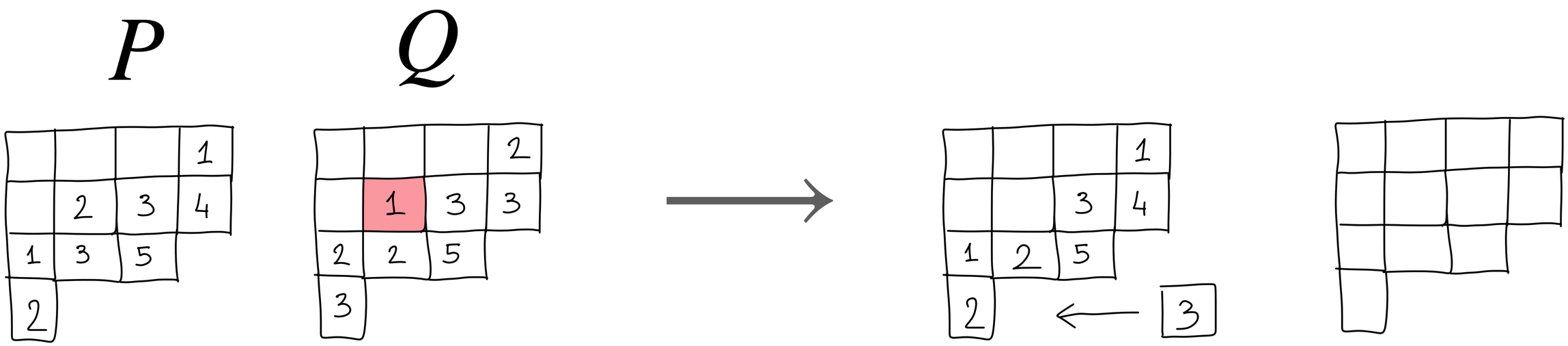

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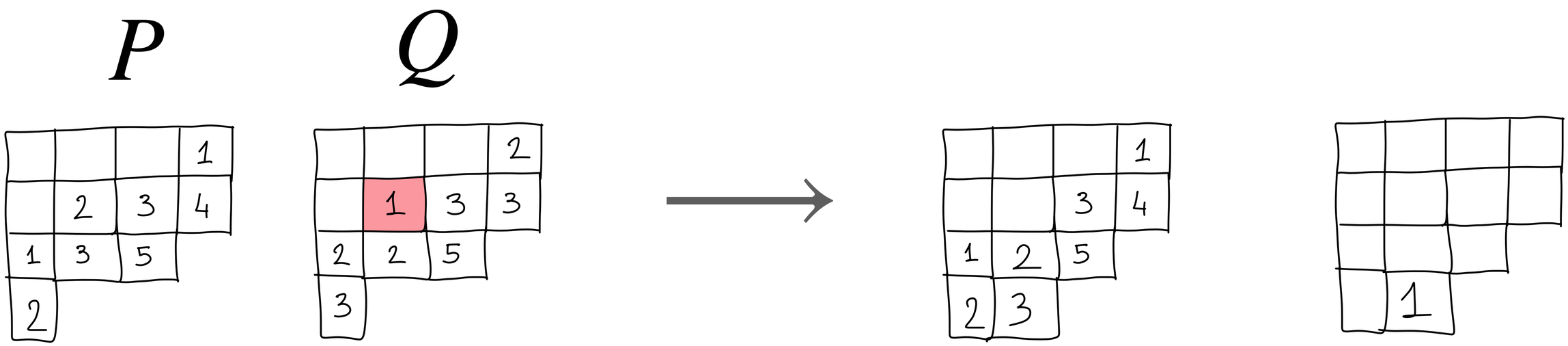

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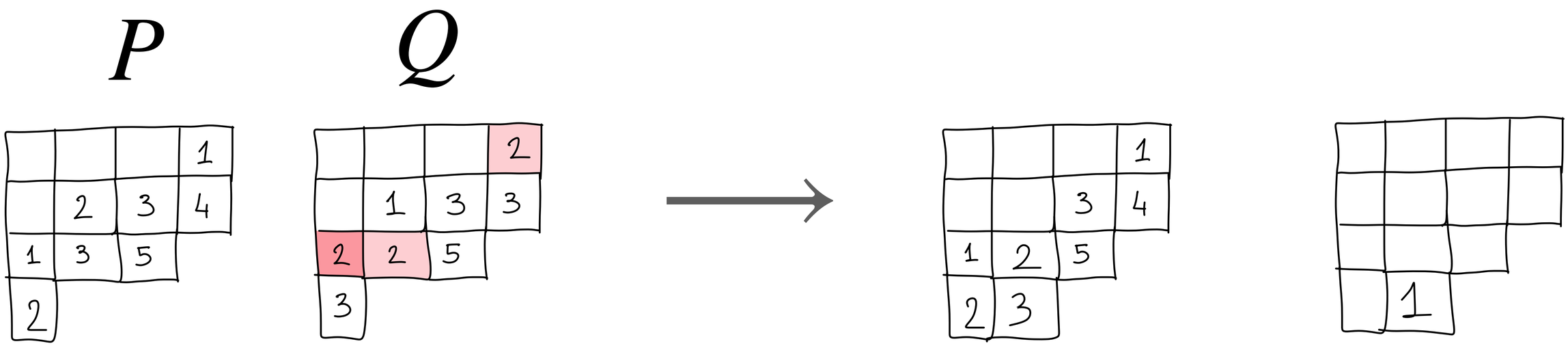

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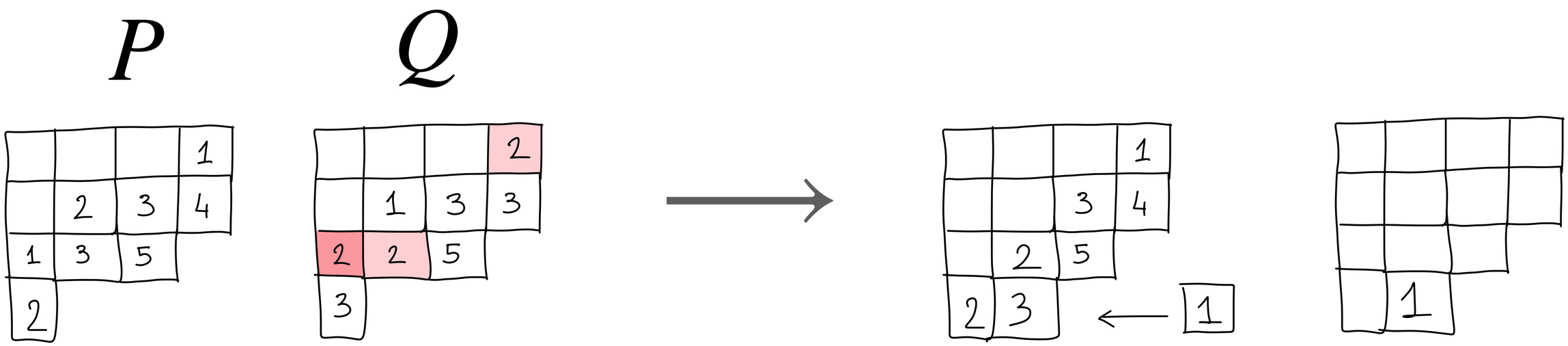

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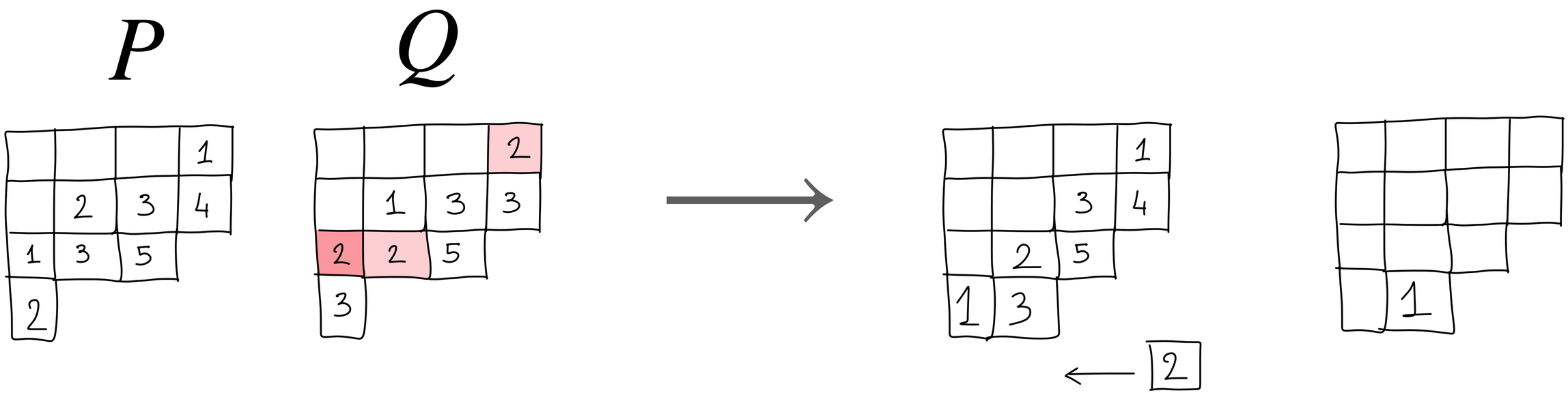

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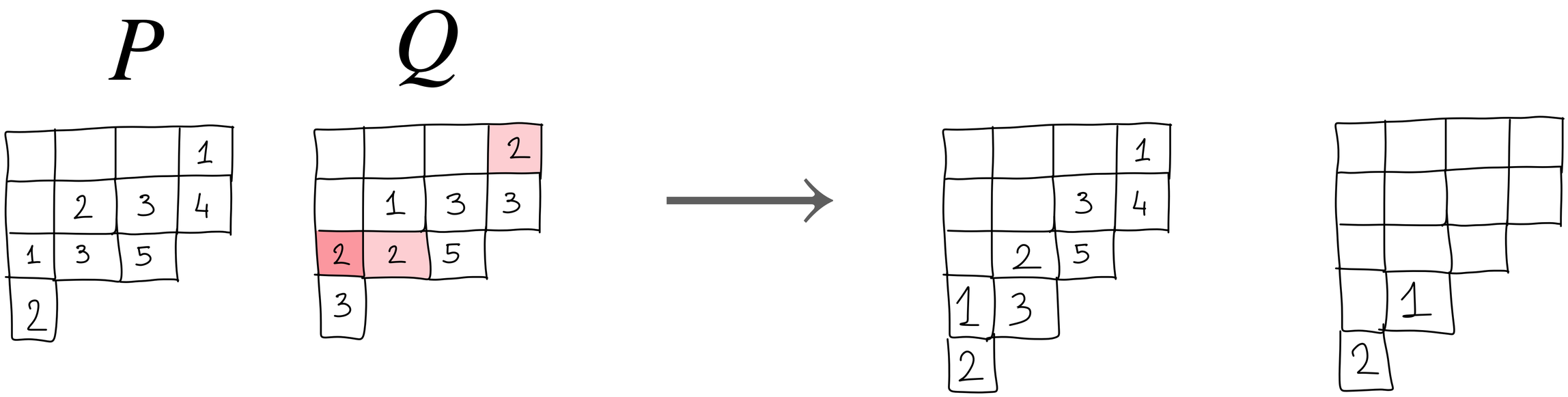

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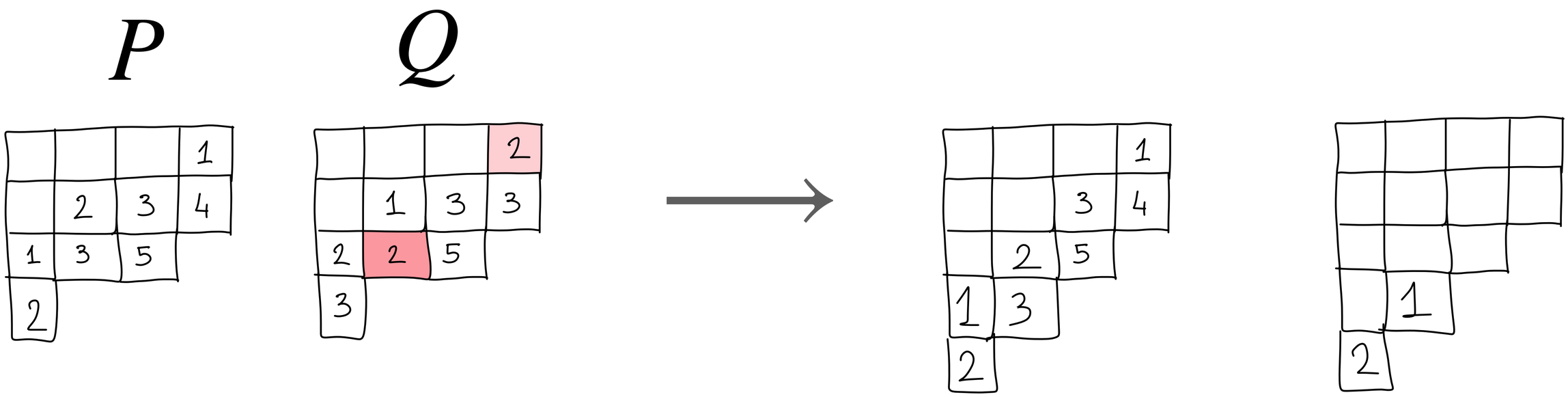

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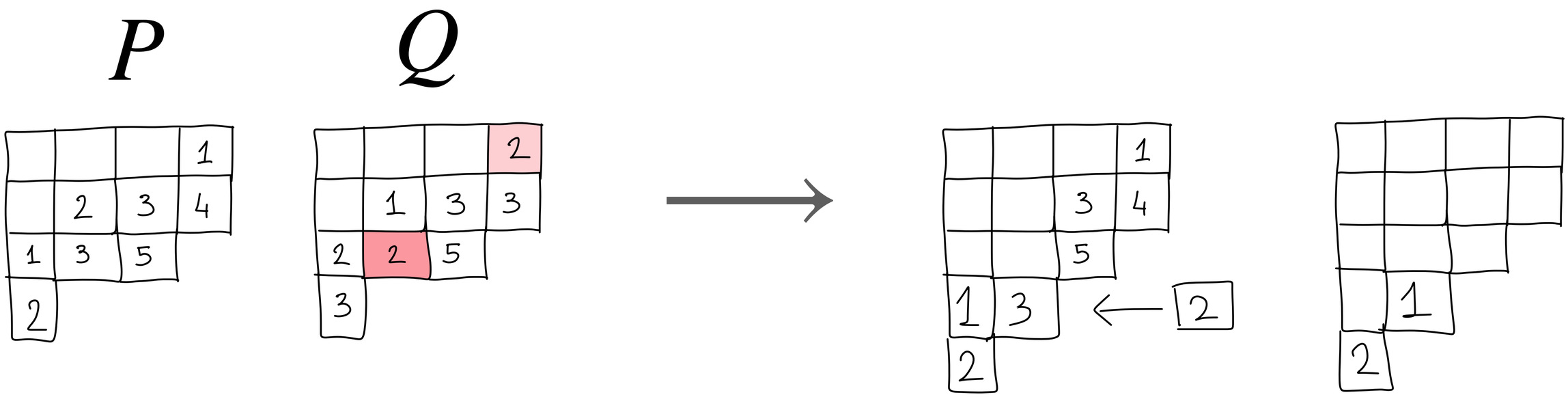

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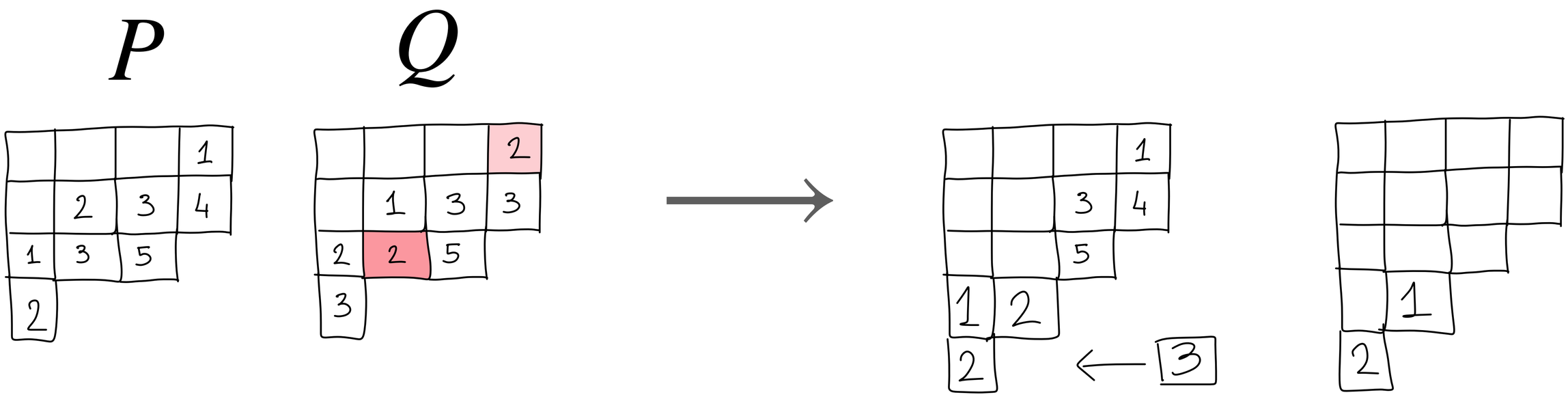

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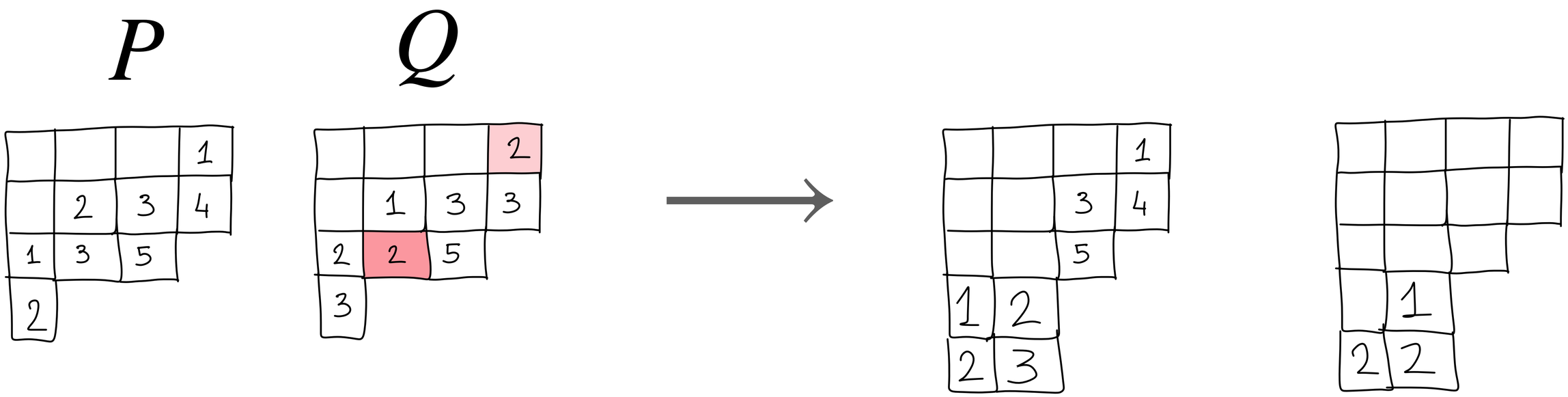

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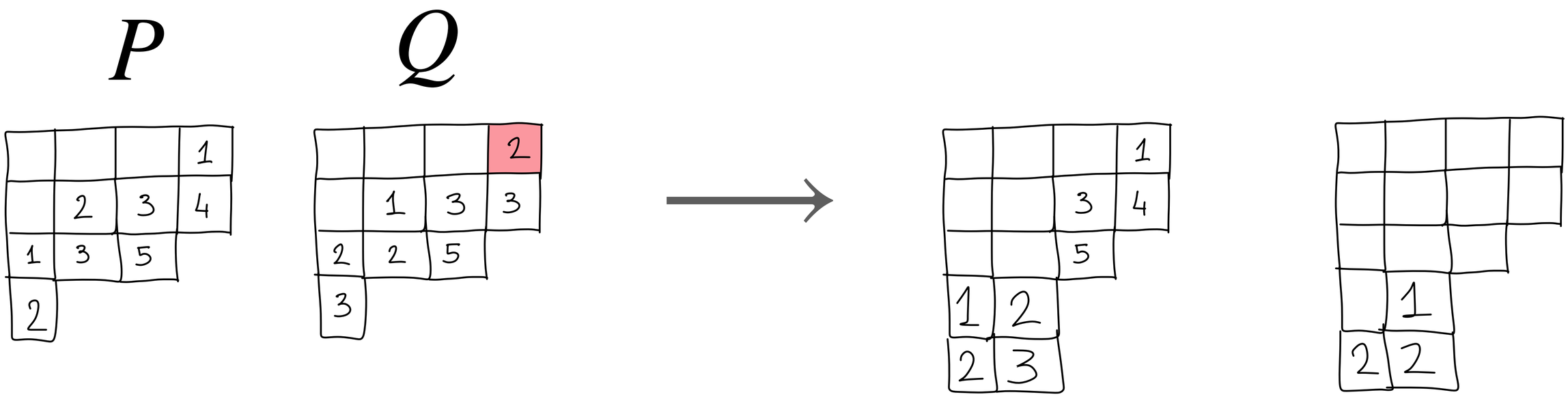
Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)



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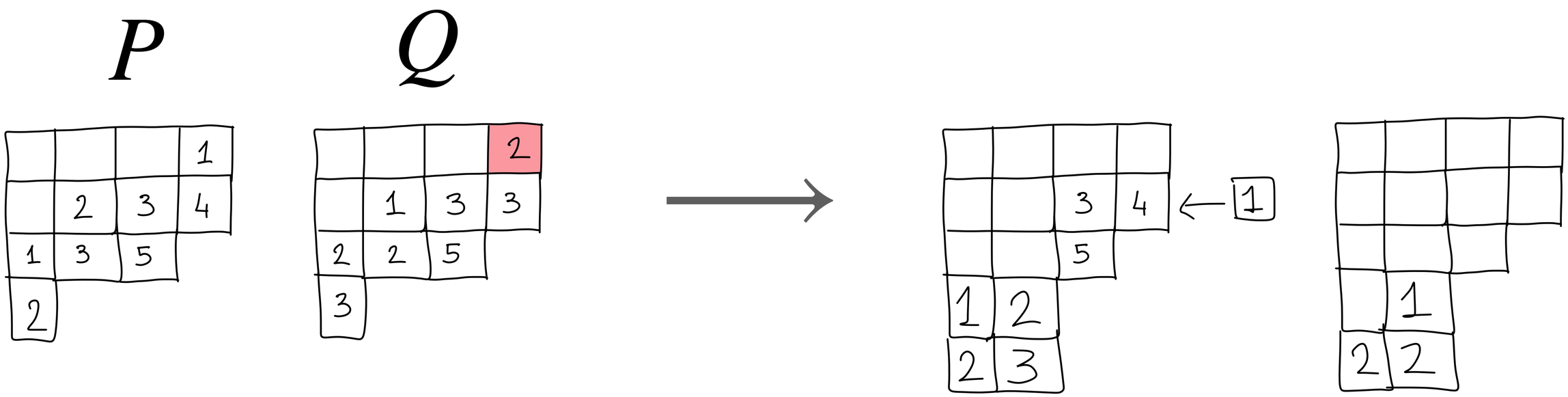

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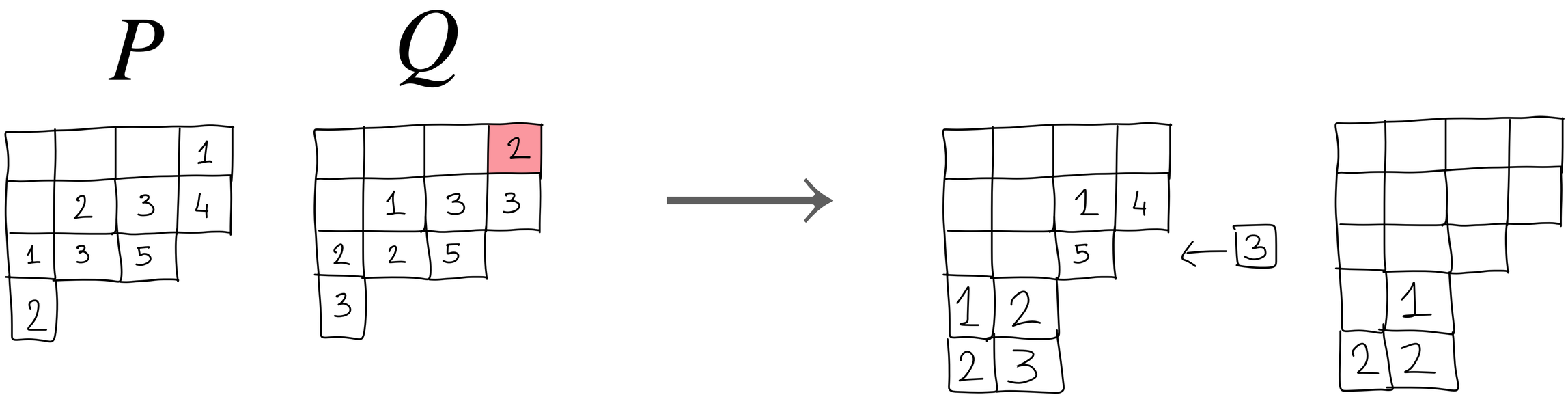

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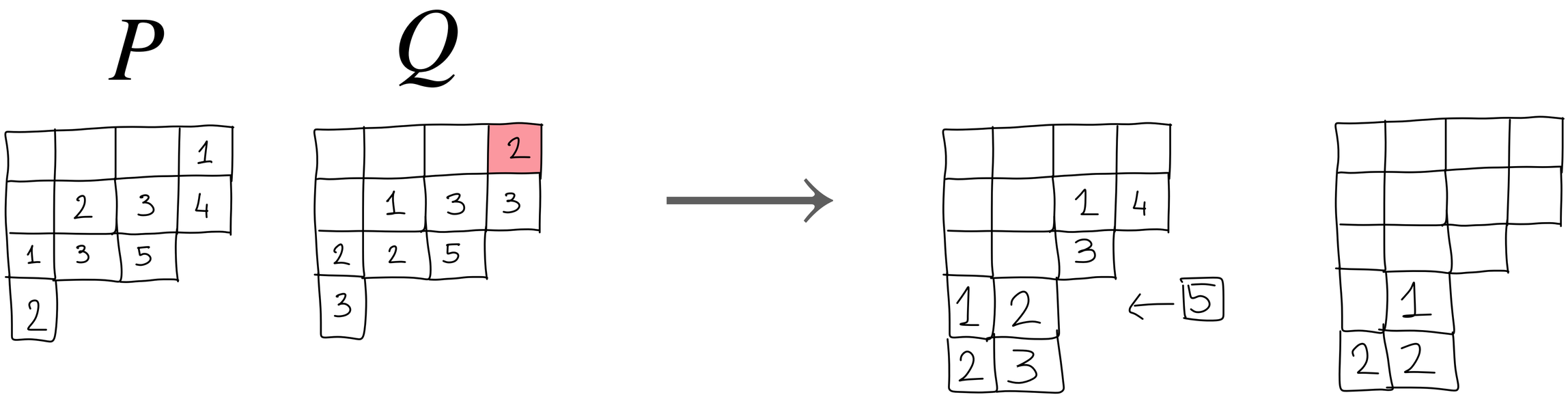

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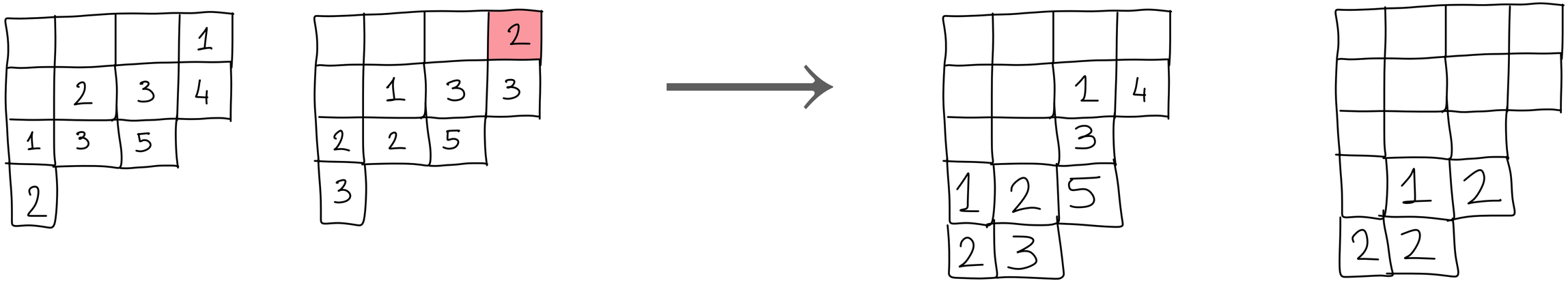

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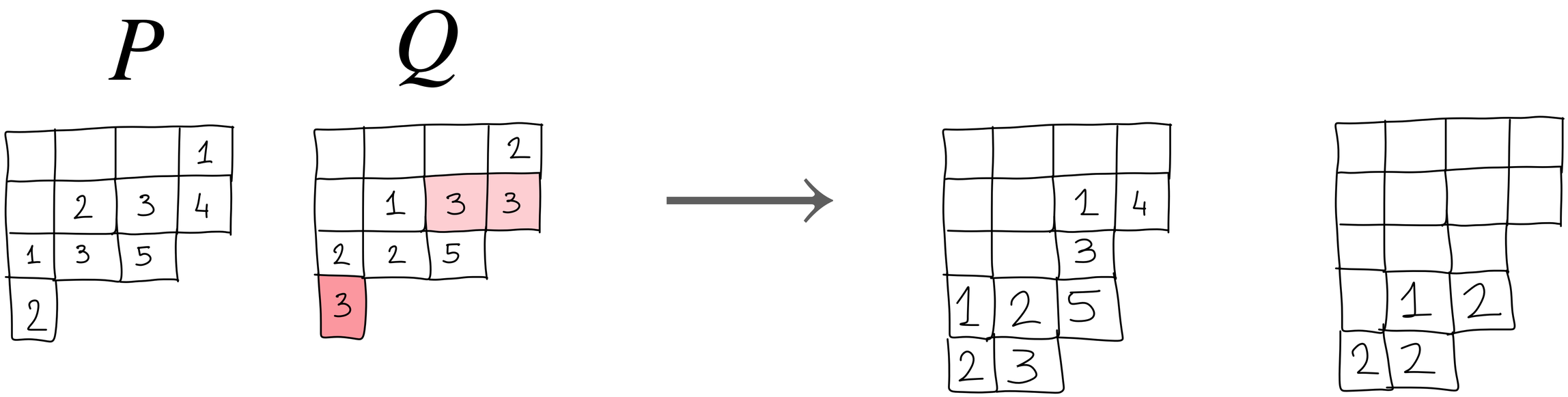

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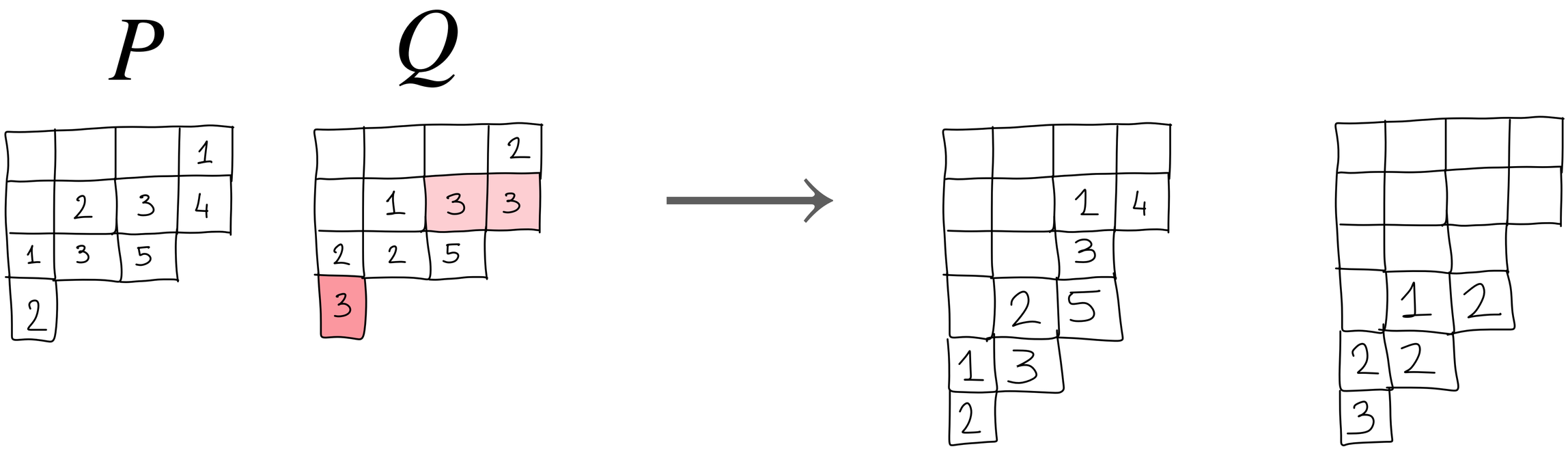

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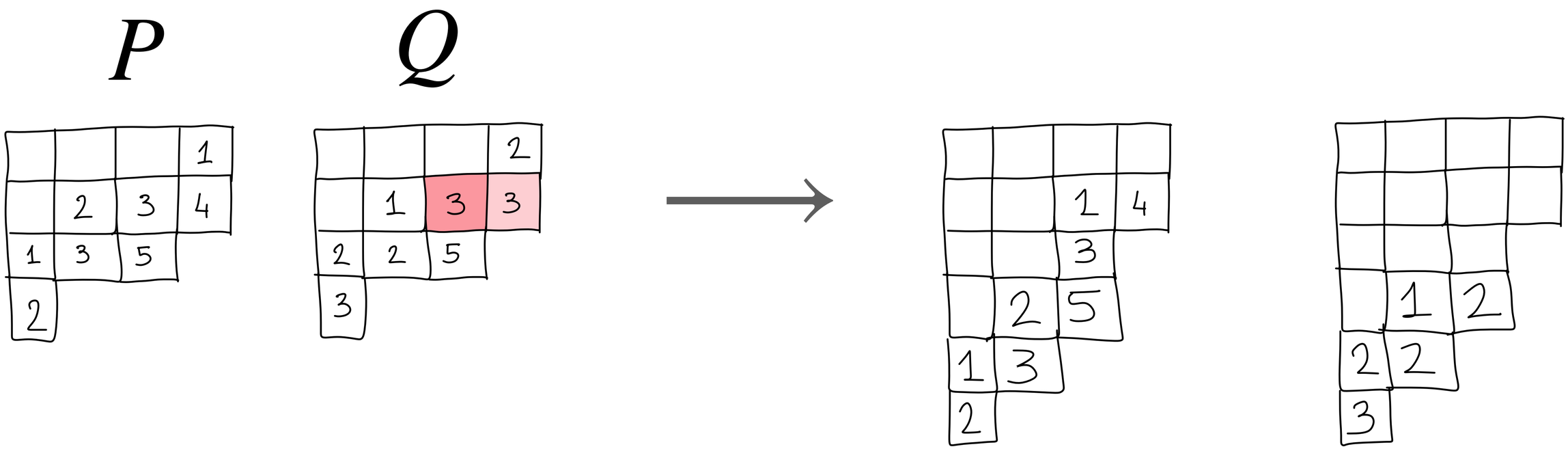

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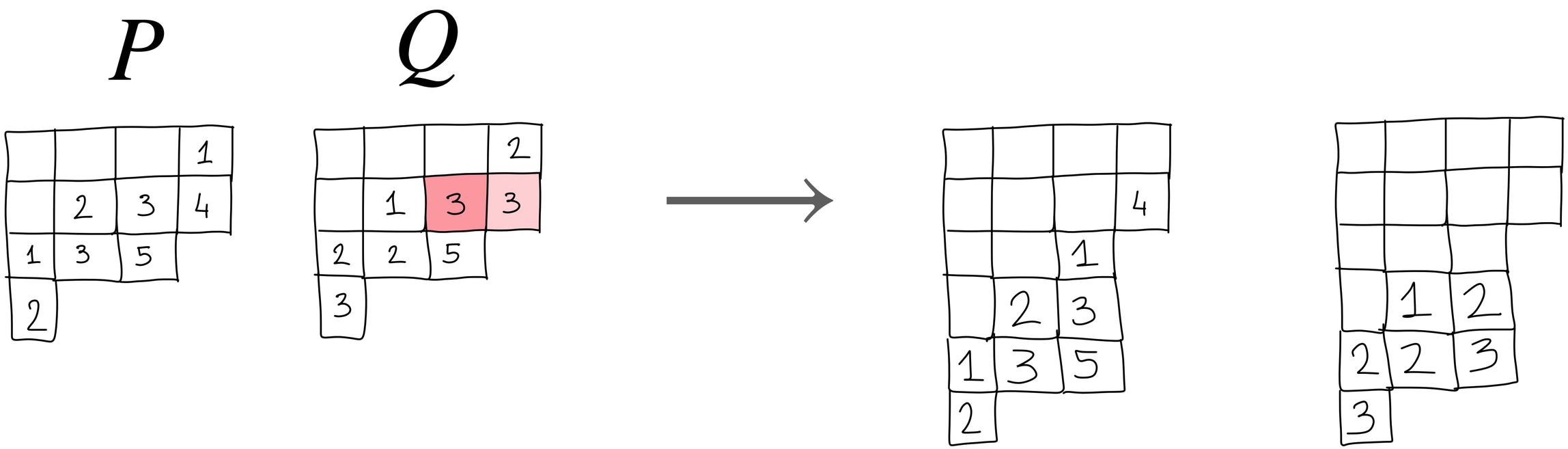

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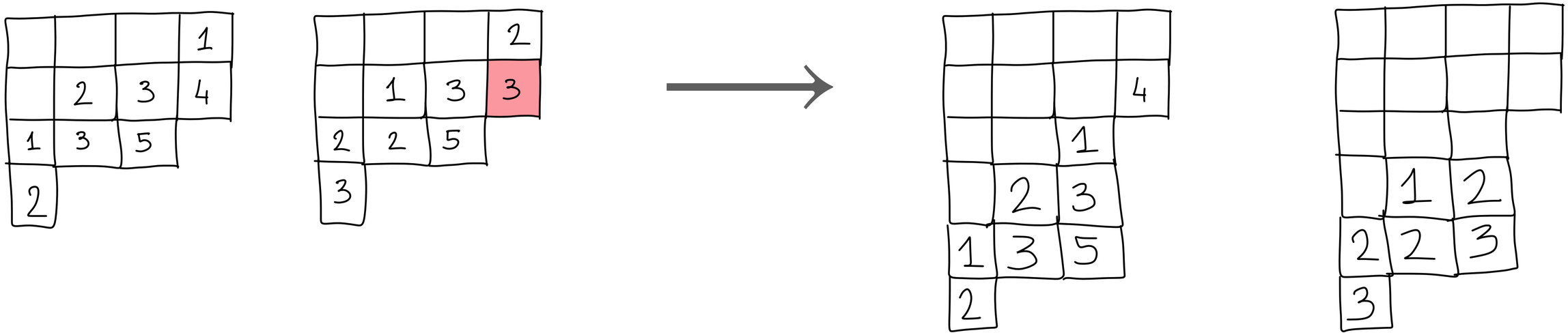
P $\qquad$ Q



- To construct (V, W) we define a dynamics on tableaux based on internal insertion [Sagan-Stanley'90]

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)

P

			1
	2	3	4
1	3	5	
2			

Q

			2
	1	3	3
2	2	5	
3			

$\longrightarrow$

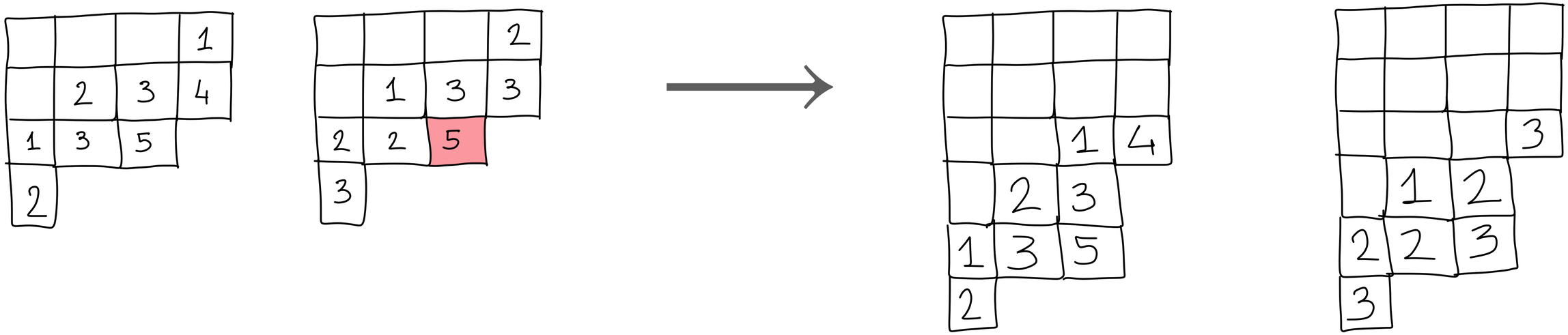
		1	4
	2	3	
1	3	5	
2			

			3
	1	2	
2	2	3	
3			

- To construct (V, W) we define a dynamics on tableaux based on internal insertion [Sagan-Stanley'90]

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)

P $\qquad$ Q



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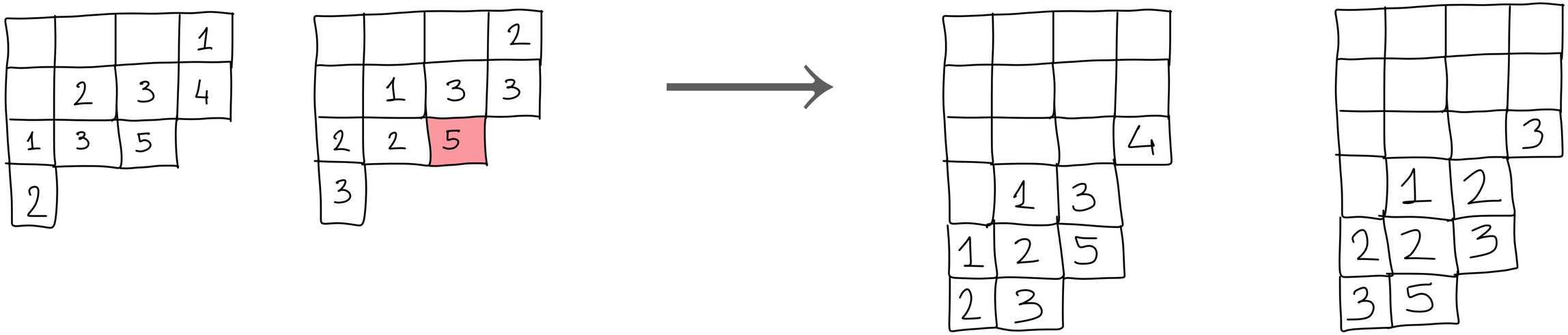
P $\qquad$ Q



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Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)

P

			1
	2	3	4
1	3	5	
2			

Q

			2
	1	3	3
2	2	5	
3			

$\xrightarrow{\text{RSK}}$

P'

			4
	1	3	
1	2	5	
2	3		

Q'

			3
	1	2	
2	2	3	
3	5		

- To construct (V, W) we define a dynamics on tableaux based on internal insertion [Sagan-Stanley'90]

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)

P

			1
	2	3	4
1	3	5	
2			

Q

			2
	1	3	3
2	2	5	
3			

$\xrightarrow{\text{RSK}}$

P'

			4
	1	3	
1	2	5	
2	3		

Q'

			3
	1	2	
2	2	3	
3	5		

- To construct (V, W) we define a dynamics on tableaux based on internal insertion [Sagan-Stanley'90]

$$(P, Q) \to (P', Q')$$

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ ✓ (sketch)

P

			1
	2	3	4
1	3	5	
2			

Q

			2
	1	3	3
2	2	5	
3			

$\xrightarrow{\text{RSK}}$

P'

			4
	1	3	
1	2	5	
2	3		

Q'

			3
	1	2	
2	2	3	
3	5		

- To construct (V, W) we define a dynamics on tableaux based on internal insertion [Sagan-Stanley'90]

$$(P, Q) \to (P', Q') \to \cdots \to (P^{(n)}, Q^{(n)})$$

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ ✓ (sketch)

P

			1
	2	3	4
1	3	5	
2			

Q

			2
	1	3	3
2	2	5	
3			

$\xrightarrow{\mathbf{RSK}^{10}}$

$P^{(10)}$

	⋮	⋮	⋮	⋮
12				4
⋮	⋮	⋮	⋮	
22			3	
23		1	5	
24		2		
⋮	⋮			
31	1			
32	2			
33	3			

$Q^{(10)}$

	⋮	⋮	⋮	⋮
12				3
⋮	⋮	⋮	⋮	
22			2	
23		2	3	
24		5		
⋮	⋮			
31	1			
32	2			
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Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ ✓ (sketch)

P

			1
	2	3	4
1	3	5	
2			

Q

			2
	1	3	3
2	2	5	
3			

$\xrightarrow{\mathbf{RSK}^{10}}$

$P^{(10)}$

	⋮	⋮	⋮	⋮
12				4
⋮	⋮	⋮	⋮	
22			3	
23		1	5	
24		2		
⋮	⋮			
31	1			
32	2			
33	3			

$Q^{(10)}$

	⋮	⋮	⋮	⋮
12				3
⋮	⋮	⋮	⋮	
22			2	
23		2	3	
24		5		
⋮	⋮			
31	1			
32	2			
33	3			

- To construct (V, W) we define a dynamics on tableaux based on internal insertion [Sagan-Stanley'90]

$$(P, Q) \to (P', Q') \to \cdots \to (P^{(n)}, Q^{(n)}) \longrightarrow (V, W)$$

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)

P

			1
	2	3	4
1	3	5	
2			

Q

			2
	1	3	3
2	2	5	
3			

$\xrightarrow{\mathbf{RSK}^{10}}$

$P^{(10)}$

	⋮	⋮	⋮	⋮
12				4
⋮	⋮	⋮	⋮	
22			3	
23		1	5	
24		2		
⋮	⋮			
31	1			
32	2			
33	3			

$Q^{(10)}$

	⋮	⋮	⋮	⋮
12				3
⋮	⋮	⋮	⋮	
22			2	
23		2	3	
24		5		
⋮	⋮			
31	1			
32	2			
33	3			

- To construct (V, W) we define a dynamics on tableaux based on internal insertion [Sagan-Stanley'90]

$$(P, Q) \to (P', Q') \to \cdots \to (P^{(n)}, Q^{(n)}) \longrightarrow (V, W) =$$

1	1	3	4
2	2	5	
3			

1	2	2	3
2	5	3	
3			

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)

P

			1
	2	3	4
1	3	5	
2			

Q

			2
	1	3	3
2	2	5	
3			

$\xrightarrow{\mathbf{RSK}^{10}}$

$P^{(10)}$

	⋮	⋮	⋮	⋮
12				4
⋮	⋮	⋮	⋮	
22			3	
23		1	5	
24		2		
⋮	⋮			
31	1			
32	2			
33	3			

$Q^{(10)}$

	⋮	⋮	⋮	⋮
12				3
⋮	⋮	⋮	⋮	
22			2	
23		2	3	
24		5		
⋮	⋮			
31	1			
32	2			
33	3			

- It remains to construct κ.

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)

P

			1
	2	3	4
1	3	5	
2			

Q

			2
	1	3	3
2	2	5	
3			

$\xrightarrow{\mathbf{RSK}^{10}}$

$P^{(10)}$

	⋮	⋮	⋮	⋮
12				4
⋮	⋮	⋮	⋮	
22			3	
23		1	5	
24		2		
⋮	⋮			
31	1			
32	2			
33	3			

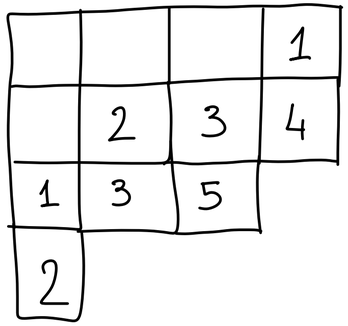
$Q^{(10)}$

	⋮	⋮	⋮	⋮
12				3
⋮	⋮	⋮	⋮	
22			2	
23		2	3	
24		5		
⋮	⋮			
31	1			
32	2			
33	3			

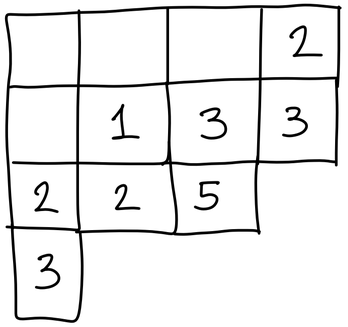
- It remains to construct κ.
- $\mu =$ shape of (V, W) $\quad \tilde{\lambda}/\tilde{\rho} =$ shape of $(P^{(n)}, Q^{(n)})$ for n large

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)

P

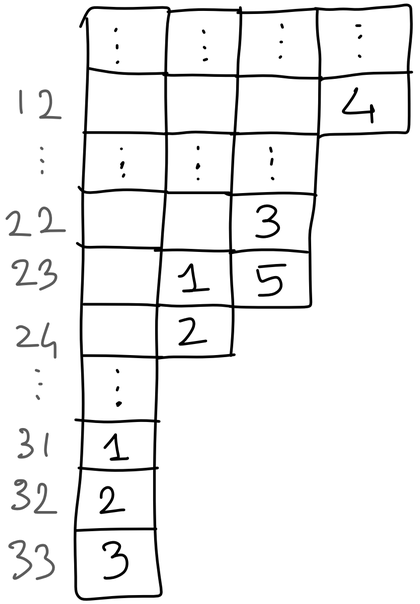


Q

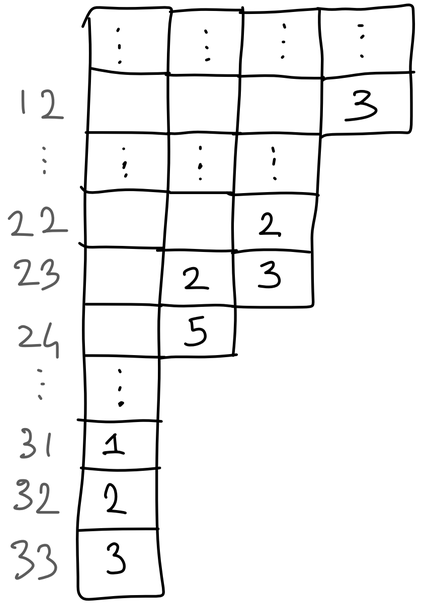


$\xrightarrow{\mathbf{RSK}^{10}}$

$P^{(10)}$



$Q^{(10)}$



- It remains to construct κ.
- $\mu =$ shape of (V, W) $\quad \tilde{\lambda}/\tilde{\rho} =$ shape of $(P^{(n)}, Q^{(n)})$ for n large
- Define $\tau_i = \tilde{\rho}_i' - n \times \mu_i'$

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)

P

			1
	2	3	4
1	3	5	
2			

Q

			2
	1	3	3
2	2	5	
3			

$\xrightarrow{\mathbf{RSK}^{10}}$

$P^{(10)}$

	⋮	⋮	⋮	⋮
12				4
⋮	⋮	⋮	⋮	
22			3	
23		1	5	
24		2		
⋮	⋮			
31	1			
32	2			
33	3			

$Q^{(10)}$

	⋮	⋮	⋮	⋮
12				3
⋮	⋮	⋮	⋮	
22			2	
23		2	3	
24		5		
⋮	⋮			
31	1			
32	2			
33	3			

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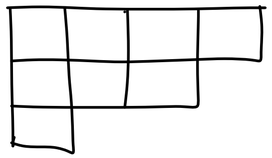
- Define $\tau_i = \tilde{\rho}'_i - n \times \mu'_i$

$\tau_1 = 0 = 30 - 10 \times 3$

$\mu =$



Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)

P

			1
	2	3	4
1	3	5	
2			

Q

			2
	1	3	3
2	2	5	
3			

$\xrightarrow{\mathbf{RSK}^{10}}$

$P^{(10)}$

	⋮	⋮	⋮	⋮
12				4
⋮	⋮	⋮	⋮	
22			3	
23		1	5	
24		2		
⋮	⋮			
31	1			
32	2			
33	3			

$Q^{(10)}$

	⋮	⋮	⋮	⋮
12				3
⋮	⋮	⋮	⋮	
22			2	
23		2	3	
24		5		
⋮	⋮			
31	1			
32	2			
33	3			

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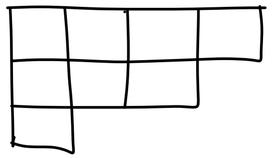
$\mu =$



$\tau_1 = 0 = 30 - 10 \times 3$

$\tau_2 = 2 = 22 - 10 \times 2$

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)

P

			1
	2	3	4
1	3	5	
2			

Q

			2
	1	3	3
2	2	5	
3			

$\xrightarrow{\mathbf{RSK}^{10}}$

$P^{(10)}$

	⋮	⋮	⋮	⋮
12				4
⋮	⋮	⋮	⋮	
22			3	
23		1	5	
24		2		
⋮	⋮			
31	1			
32	2			
33	3			

$Q^{(10)}$

	⋮	⋮	⋮	⋮
12				3
⋮	⋮	⋮	⋮	
22			2	
23		2	3	
24		5		
⋮	⋮			
31	1			
32	2			
33	3			

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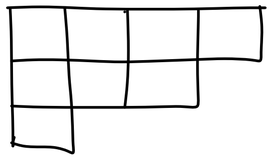

$$\tau_1 = 0 = 30 - 10 \times 3$$
$$\tau_2 = 2 = 22 - 10 \times 2$$
$$\tau_3 = 1 = 21 - 10 \times 2$$

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)

P

			1
	2	3	4
1	3	5	
2			

Q

			2
	1	3	3
2	2	5	
3			

$\xrightarrow{\mathbf{RSK}^{10}}$

$P^{(10)}$

	⋮	⋮	⋮	⋮
12				4
⋮	⋮	⋮	⋮	
22			3	
23		1	5	
24		2		
⋮	⋮			
31	1			
32	2			
33	3			

$Q^{(10)}$

	⋮	⋮	⋮	⋮
12				3
⋮	⋮	⋮	⋮	
22			2	
23		2	3	
24		5		
⋮	⋮			
31	1			
32	2			
33	3			

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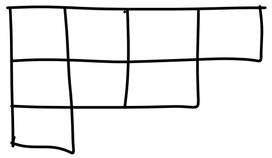
- Define $\tau_i = \tilde{\rho}'_i - n \times \mu'_i$

$\mu =$



$$\tau_1 = 0 = 30 - 10 \times 3$$
$$\tau_2 = 2 = 22 - 10 \times 2$$
$$\tau_3 = 1 = 21 - 10 \times 2$$
$$\tau_4 = 1 = 11 - 10 \times 1$$

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)

P

			1
	2	3	4
1	3	5	
2			

Q

			2
	1	3	3
2	2	5	
3			

$\longrightarrow$

V

1	1	3	4
2	2	5	
3			

W

1	2	2	3
2	5	3	
3			

τ

$(0,2,1,1)$

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)

P

			1
	2	3	4
1	3	5	
2			

Q

			2
	1	3	3
2	2	5	
3			

$\longrightarrow$

V

1	1	3	4
2	2	5	
3			

W

1	2	2	3
2	5	3	
3			

τ

$(0,2,1,1)$

- $(P, Q) \leftrightarrow (V, W; \tau)$ is a bijection, but τ depends on (V, W)!

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)

P

			1
	2	3	4
1	3	5	
2			

Q

			2
	1	3	3
2	2	5	
3			

$\longrightarrow$

V

1	1	3	4
2	2	5	
3			

W

1	2	2	3
2	5	3	
3			

τ

$(0,2,1,1)$

- $(P, Q) \leftrightarrow (V, W; \tau)$ is a bijection, but τ depends on (V, W)!

LEMMA (IMS'21)

$$\tau_i = \kappa_i + \mathscr{H}_i(V) + \mathscr{H}_i(W) \qquad (\triangle)$$

$\mathscr{H}_i$ = local energy function

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)

P

			1
	2	3	4
1	3	5	
2			

Q

			2
	1	3	3
2	2	5	
3			

$\longrightarrow$

V

1	1	3	4
2	2	5	
3			

W

1	2	2	3
2	5	3	
3			

τ

$(0,2,1,1)$

- $(P, Q) \leftrightarrow (V, W; \tau)$ is a bijection, but τ depends on (V, W)!

LEMMA (IMS'21)

$$\tau_i = \kappa_i + \mathscr{H}_i(V) + \mathscr{H}_i(W) \qquad (\triangle)$$

$\mathscr{H}_i$ = local energy function

Comments on $(\triangle)$:

- Shows deep connections between theories of skew tableaux and theories of discrete integrable systems, Kashiwara crystals, …
- $\mathscr{H}_i$ appear in the description of *phase shift* in the $\widehat{\mathfrak{sl}}_n$ Box and Ball System
- This is because the skew RSK dynamics is a generalization of the Box and Ball Systems

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)

P

			1
	2	3	4
1	3	5	
2			

Q

			2
	1	3	3
2	2	5	
3			

$\xrightarrow{\text{RSK}}$

V

1	1	3	4
2	2	5	
3			

W

1	2	2	3
2	5	3	
3			

τ

$(0,2,1,1)$

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- To prove energy ($\mathscr{H}$) formulas we need to study symmetries of the skew RSK
- Classical RSK commutes with $\mathfrak{sl}_n$ Kashiwara operators
- Energy $\mathscr{H}$ comes from affine ($\widehat{\mathfrak{sl}}_n$) crystals

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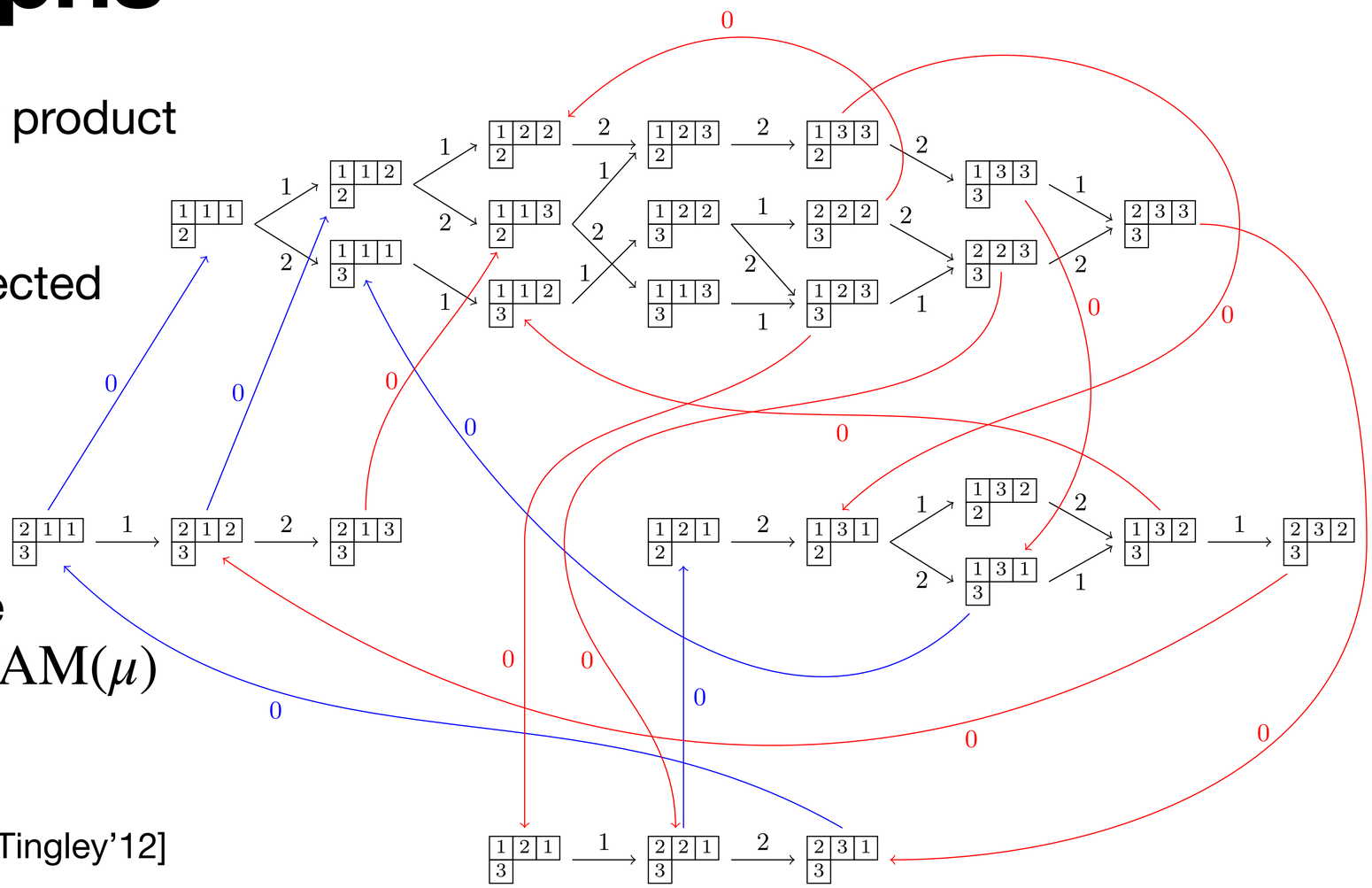
- IDEA : equip (P, Q) and (V, W) of $\widehat{\mathfrak{sl}}_n$ - bicrystal structure preserved by RSK

Affine crystal graphs

- Vertically strict tableaux are tensor product of KR crystals
- The crystal graph $VST(\mu)$ is connected [Kashiwara'90]
- Demazure subgraph: remove "bad" $\tilde{e}_0, \tilde{f}_0$ arrows
- For any V there exist a path on the Demazure subgraph $\mathscr{L}_V : V \mapsto \mathrm{YAM}(\mu)$
- $\mathscr{H}(V) = \#\tilde{f}_0 - \#\tilde{e}_0$ in $\mathscr{L}_V$

[Schilling-Tingley'12]



Affine Bicrystal (P, Q)

- On (V, W) bicrystal structure is product structure
- Impose $(P, Q) \mapsto (V, W)$ preserve bicrystal structure
- On (V, W) bicrystal structure is NOT product structure (nontrivial $\widetilde{e}_0, \tilde{f}_0$)
- We transport maps $\mathscr{L}_V, \mathscr{L}_W \mapsto \mathscr{L}_{P,Q}$

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Example:

$$\left(\begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & 2 & 3 & 4 \\ \hline 1 & 3 & 5 & \\ \hline 2 & & & \\ \hline \end{array} \, , \, \begin{array}{|c|c|c|c|} \hline & & & 2 \\ \hline & 1 & 3 & 3 \\ \hline 2 & 2 & 5 & \\ \hline 3 & & & \\ \hline \end{array} \right) \longrightarrow \left(\begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 & \\ \hline \end{array} \, , \, \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 & \\ \hline \end{array} \right)$$

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LEMMA (IMS'21)

$\mathscr{L}_{P,Q}$ linearizes the skew RSK map

To sum up

- $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ has the following properties
  - P and V have equal content. Same for Q and W
  - $|\rho| = \mathscr{H}(V) + \mathscr{H}(W) + |\kappa| + |\nu|$ $\qquad (\mathscr{H} = \mathscr{H}_1 + \mathscr{H}_2 + \cdots)$
  - $\lambda_1 = \mu_1 + \nu_1$

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Conclusion

- Our bijective theory gives direct connection between solvable non-free fermionic models and positive temperature free fermionic models

- Mysterious appearence of a discrete integrable system (KdV like models)

- We construct a *bijective* q-extension of the RSK correspondence Υ

- With Υ we can prove bijectively the Cauchy identities (CI) for q-Whittaker polynomials (first time)

- With Υ we can prove bijectively refinements of the CI relating q-Whittaker and skew-Schur polynomials