Integrable cellular automata, free fermions and KPZ solvable models.

Rigorous Statistical Mechanics and Related Topics
RIMS, Kyoto
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arXiv:2106.11922[math.CO]
arXiv:2106.11913[math.CO]

Plan of the talk

- **Part 1.** Presenting the problem we want to solve: mysterious connections between KPZ equation and free fermions at positive temperature

- **Part 2.** combinatorial construction of the correspondence KPZ models - free fermionic systems

Plan of the talk

- **Part 1.** Presenting the problem we want to solve: mysterious connections between KPZ equation and free fermions at positive temperature

KPZ equation

[Kardar-Parisi-Zhang '86]

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \eta$$

- $h =$ random height function $\eta =$ space-time white noise
- What is it?
- Well posedness:
  - Bertini-Giacomin '97
  - Hairer '11
  - Gubinelli-Perkowski-Imkeller '12

KPZ equation : exact solutions

$$\begin{cases} \partial_t h = \partial_x^2 h + (\partial_x h)^2 + \eta \\ h(x,0) = \log(\delta_x) \end{cases}$$

- Narrow wedge initial conditions
  [Amir-Corwin-Quastel, Calabrese-Le Doussal, Dotsenko, Sasamoto-Spohn ’11]

$$\mathbb{E}\left[\exp\left(-z e^{h(0,t)+t/12}\right)\right] = \det\left(1 - f K_{\mathrm{Airy}}\right)_{\mathscr{L}^2(\mathbb{R})}$$

$$K_{\mathrm{Airy}}(x,y) = \int_0^\infty \mathrm{Ai}(x+z)\mathrm{Ai}(y+z)dz$$

Airy Kernel

$$f(x) = \frac{1}{1 + e^{-xt^{1/3}}/z}$$

Fermi factor

KPZ equation : exact solutions

$$\begin{cases} \partial_t h = \partial_x^2 h + (\partial_x h)^2 + \eta \qquad x \in \mathbb{R}_+ \\ h(x,0) = \log(\delta_x) \\ \partial_x h(x=0,t) = A \end{cases}$$

- Narrow wedge initial conditions in half space
  [Gueudre-Le Doussal '12, Borodin-Bufetov-Corwin'16, Barraquand-Borodin-Corwin-Wheeler '11]

$$\mathbb{E}_{\mathrm{hs}}\left[\exp\left(-z e^{h(0,t)+t/12}\right)\right] = \mathrm{Pf}\left(1 - fK\right)_{\mathscr{L}^2(\mathbb{R})}$$

$K(x,y)$

2x2 matrix kernel
(Airy-like)

$$f(x) = \frac{1}{1 + e^{-xt^{1/3}}/z}$$

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KPZ equation : exact solutions

- Narrow wedge initial conditions $$\mathbb{E}\left[\exp\left(-ze^{h(0,t)+t/12}\right)\right] = \det\left(1 - fK_{\mathrm{Airy}}\right)_{\mathscr{L}^2(\mathbb{R})}$$
- Narrow wedge initial conditions in half space $$\mathbb{E}_{\mathrm{hs}}\left[\exp\left(-ze^{h(0,t)+t/12}\right)\right] = \mathrm{Pf}\left(1 - fK\right)_{\mathscr{L}^2(\mathbb{R})}$$

- Mysterious relations between free fermions and KPZ equation
- Apparent only from the solutions
- Can we establish connections between KPZ eq. and free fermions a priori?

- Solutions are obtained through Bethe Ansatz (BA)
- BA is very powerful but requires difficult calculations and (often) non-rigorous arguments
- Can we create an elementary theory to solve the KPZ eq.?

How to solve the KPZ equation with rigor?

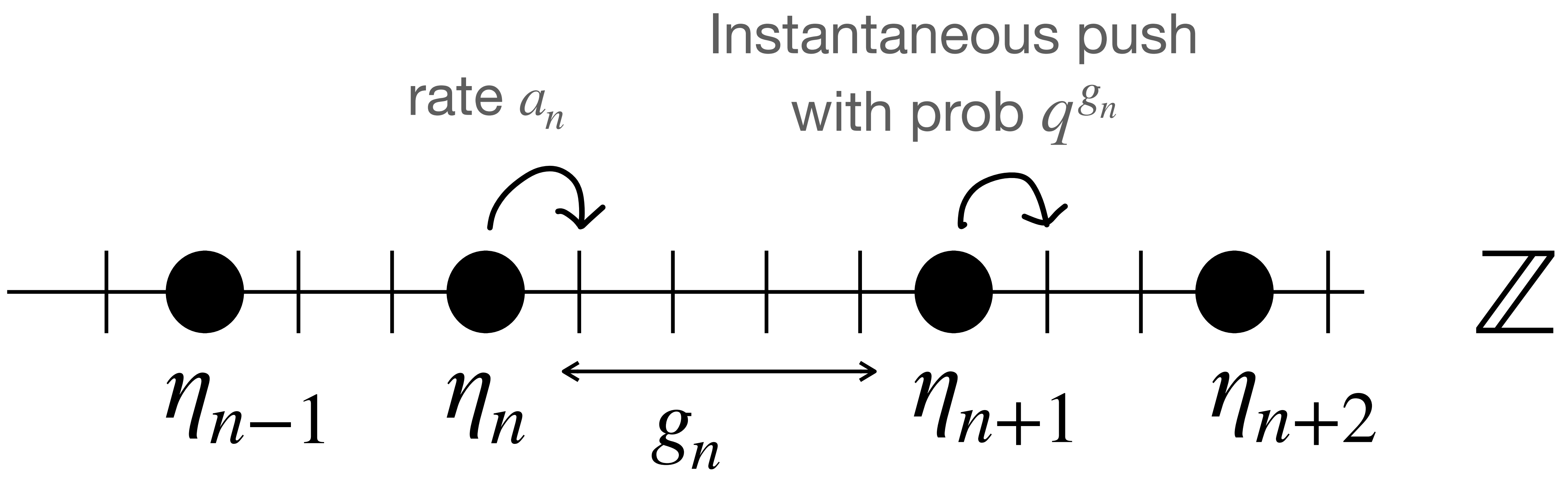
- KPZ is very irregular, also moments diverge fast!
- KPZ eq. can be approximated by discrete solvable models
  - Solvable polymer models (e.g. Log-Gamma polymers)
  - Solvable particle systems (ASEP, q-TASEP)
- Solvable models depend on a temperature parameter $q \in (0,1)$.

$$h^{(q)}(x,t) \xrightarrow[q \to 1]{} h^{KPZ}(x,t)$$

- …how to solve discrete models?

Solvable KPZ models and symmetric functions

- Solvable models in KPZ class are deeply connected to special symmetric functions from rep. theory
- Typical model: *q-Push TASEP* [Borodin-Petrov '12]



- Typical feature: Assume step initial conditions $\quad \eta_n(t)_{|t=0} = n$

$$\mathbb{P}(\eta_n(t) - n = k) = \sum_{\mu_1 = k} \frac{b_\mu \mathscr{P}_\mu(a) \mathscr{P}_\mu(b_t)}{Z^q_{a,b_t}}$$

q-Whittaker measure [Borodin-Corwin '11]

$\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq 0)$

Solvable KPZ models and symmetric functions

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- KPZ behavior captured by marginal μ_1 of the *q*-Whittaker measure
- Formulas for η_n or μ_1 are obtained through various techniques (Macdonald operators, Markov duality, Bethe Ansatz, Hecke algebras, Fock space…)
- Fredholm determinant formulas for μ_1 available [Tracy-Widom, Borodin-Corwin, Borodin-Corwin-Sasamoto] but no relation with free fermions.

- **GOAL** : relate μ_1 with natural statistics of a system of free fermions

Positive temperature Free Fermions

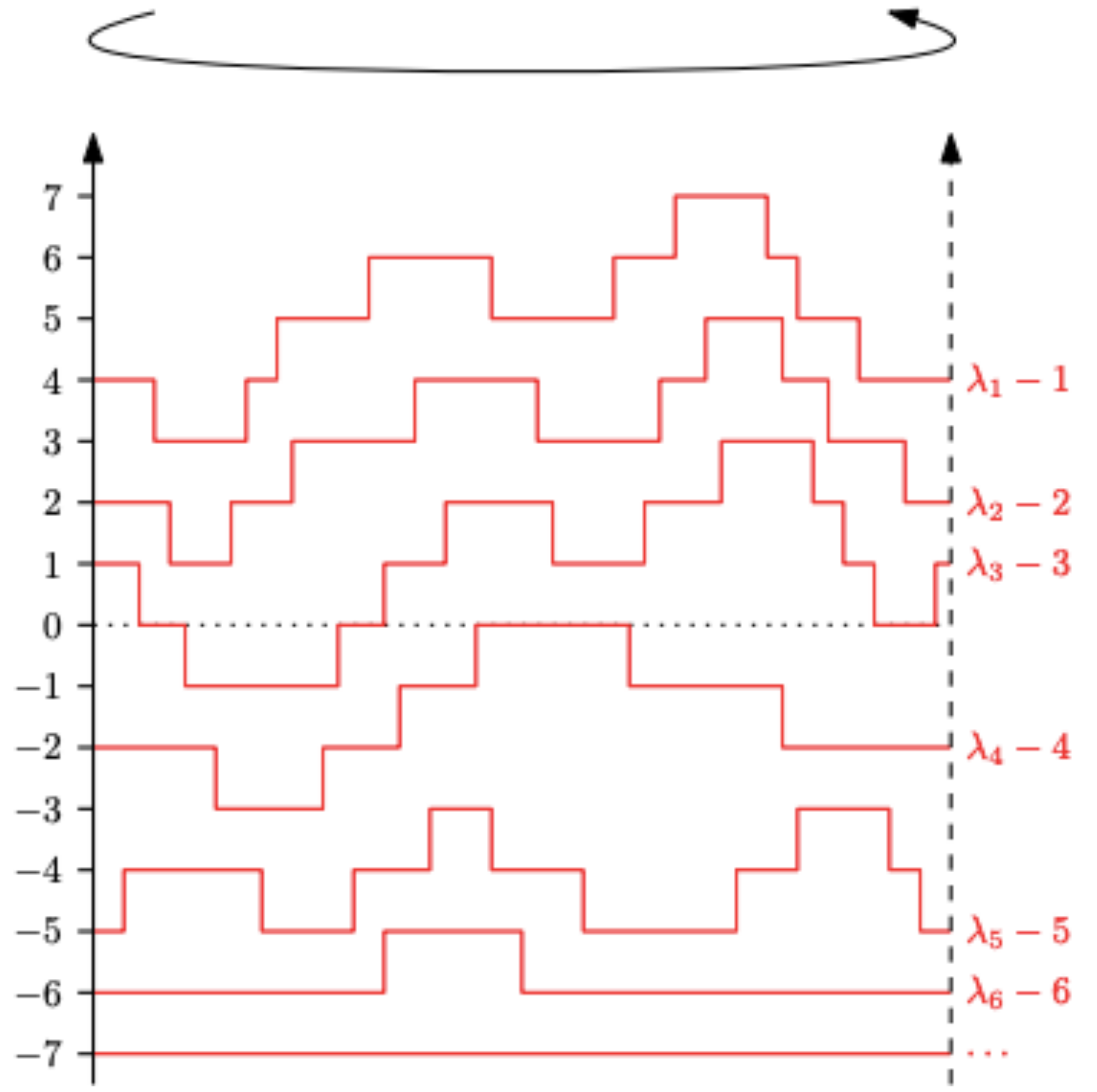
- Periodic Schur measure [Borodin '06]

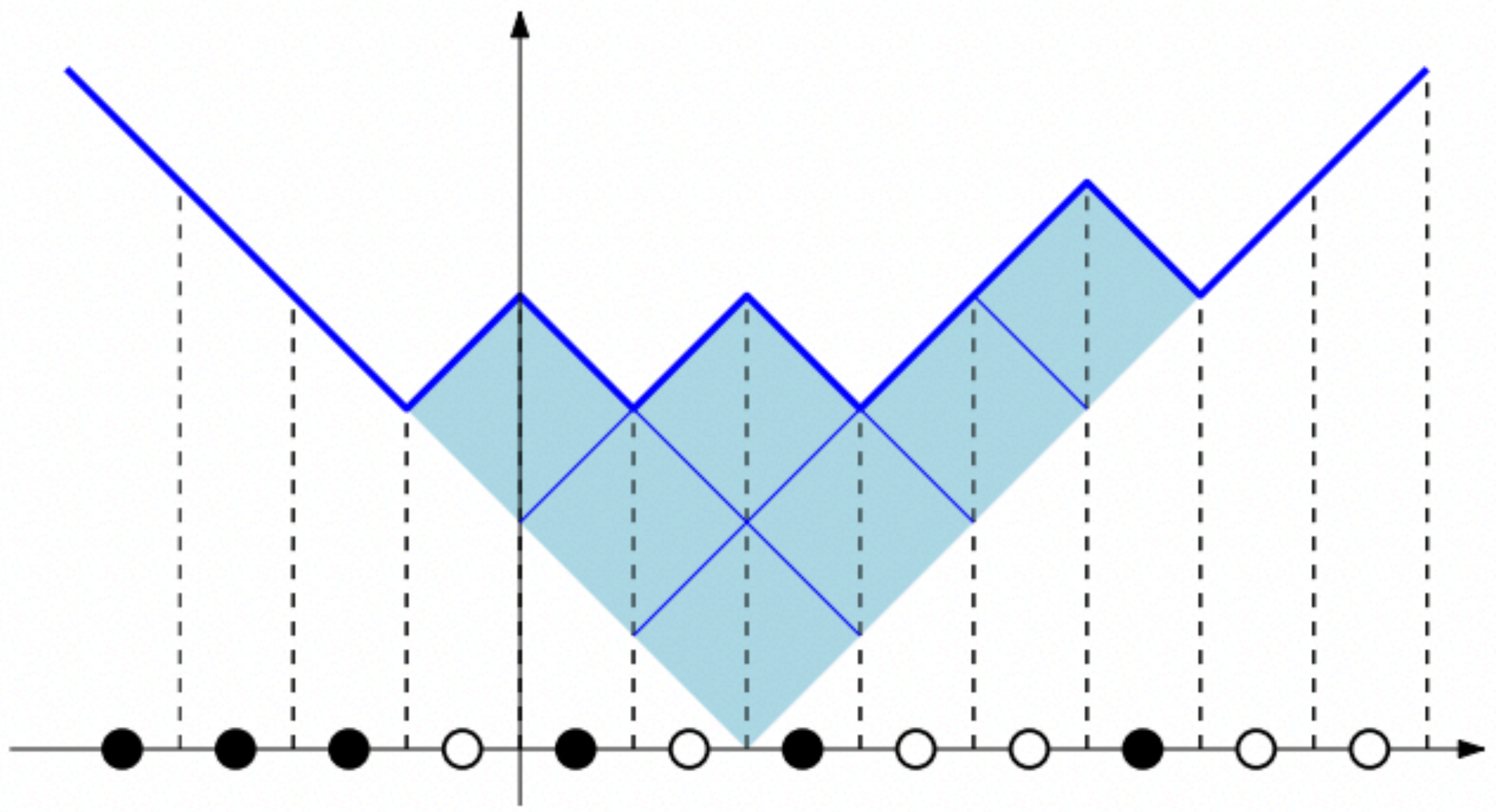
$$\mathbb{P}(\lambda) = \frac{1}{\tilde{Z}^q_{X,Y}} \sum_{\rho} q^{|\rho|} s_{\lambda/\rho}(a) s_{\lambda/\rho}(b)$$

- $\mathbb{P}(S = k) \propto q^{k^2/2} t^k$ for $k \in \mathbb{Z}$ independent of λ

- $(\lambda_1 + S, \lambda_2 + S, \lambda_3 + S, \ldots)$ is a determinantal point process

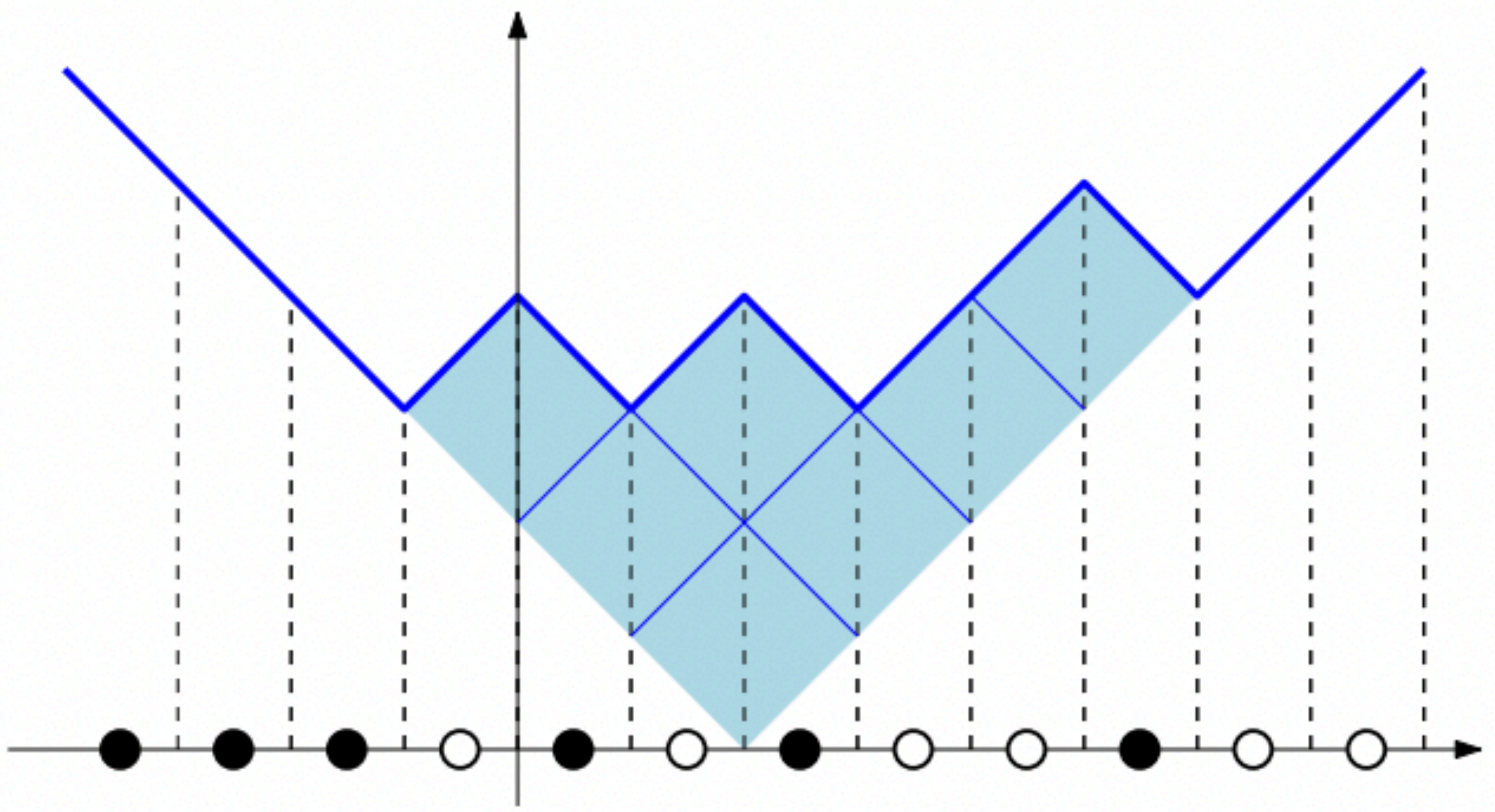
- Free fermionic nature of the PSM + edge scaling asymptotics studied in [Betea-Bouttier]





Figures from [Betea-Bouttier]

Positive temperature Free Fermions



$$\mathbb{P}(\lambda_1 + S < s) = \det(1 + K)_{\ell^2\{s,s+1,\dots\}}$$

- $(\lambda_1 + S,\ \lambda_2 + S,\ \lambda_3 + S, \dots)$ is a determinantal point process with correlation kernel

$$K(x,y) = \frac{1}{(2\pi \mathrm{i})^2} \oint_{|z|=r} \frac{dz}{z^{x+1}} \oint_{|w|=r'} \frac{dw}{w^{-y+1}} \frac{F(z)}{F(w)} \kappa(z,w),$$

$$F(z) = \prod_{i \geq 1} \frac{(b_i/z;q)_\infty}{(a_i z;q)_\infty} \qquad \kappa(z,w) = \sqrt{\frac{w}{z}} \frac{(q;q)_\infty^2}{(z/w, qw/z;q)_\infty} \frac{\vartheta_3(\zeta z/w;q)}{\vartheta_3(\zeta;q)} \qquad (z;q)_\infty = \prod_{\ell \geq 0} (1 - q^\ell z)$$

KPZ solvable models

$$\mu \sim \frac{b_\mu \mathscr{P}_\mu(x)\mathscr{P}_\mu(y)}{Z^q_{x,y}}$$

Positive temperature free fermions

$$\lambda \sim \frac{1}{\tilde{Z}^q_{X,Y}} \sum_\rho q^{|\rho|} s_{\lambda/\rho}(x) s_{\lambda/\rho}(y)$$

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THEOREM (Imamura-M.-Sasamoto '21)

$$\mu_1 + \chi \overset{\mathscr{D}}{=} \lambda_1 \qquad (\star)$$

χ independent of μ_1 and $\mathbb{P}(\chi = n) = q^n (q^{n+1}; q)_\infty$

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Comments on ($\star$):

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- suggests a new paradigm to solve models
- reveals combinatorial properties between Schur polynomials and q-Whittaker polynomials

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More Comments on ($\star$):

- A similar equivalence holds also between KPZ models in half space and other models of Free fermions (studied by [Betea-Bouttier-Nejjar-Vuletic'17])

- Earlier results relating KPZ models and free fermions:
  - [Dean-Le Doussal-Majumdar-Schehr'15]
  - [Borodin'16],[Borodin-Gorin'16],[Borodin-Ohlshanki'16],[Borodin-Corwin-Barraquand-Wheeler'17]

- In previous results the relation is deduced from comparison of formulas.

- We prove the correspondence a priori and use it to derive solution of KPZ models!

Plan of the talk

- **Part 2.** combinatorial construction of the correspondence KPZ models - free fermionic systems

- We prove ($\star$) combinatorially, developing a (bijective!) *q*-extension of the RSK.

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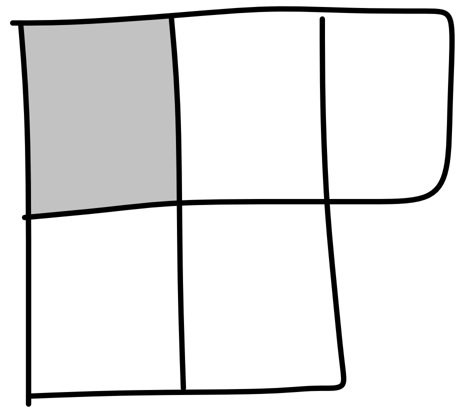
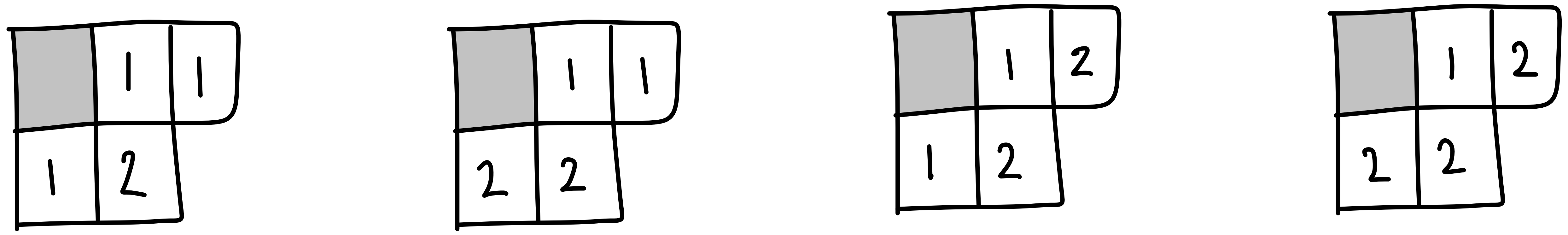
$\lambda/\rho =$ 



$$s_{\lambda/\rho}(x_1, x_2) = x_1^3 x_2 + x_1^2 x_2^2 + x_1^2 x_2^2 + x_1 x_2^3$$

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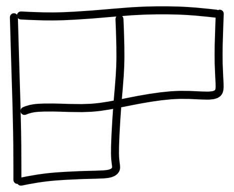
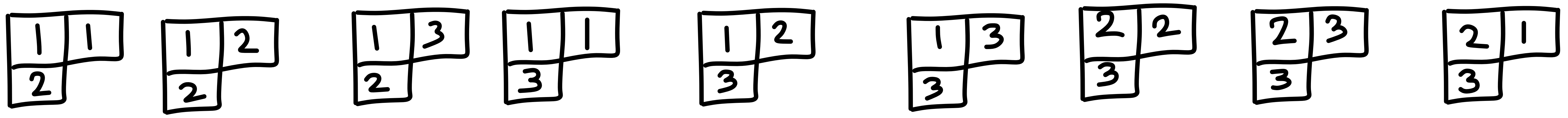
- **Combinatorial formulas**

    - $s_{\lambda/\rho}(x) = \sum_{T \in SST(\lambda/\rho)} x^T$ sum over semi-standard tableaux

    - $\mathscr{P}_\mu(x;q) = \sum_{V \in VST(\mu)} q^{\mathscr{H}(V)} x^V$ sum over “vertically strict tableaux”

      $\mathscr{H}$ = intrinsic energy

Example : $\mu =$  $n = 3$

$V =$ 

$\mathscr{H} =$ 0 0 0 0 0 0 0 0 1

$$\mathscr{P}_\mu(x;q) = x_1^2x_2 + x_1x_2^2 + x_1x_2x_3 + x_1^2x_3 + x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + qx_1x_2x_3$$

Cauchy Identities

- $$\sum_{\lambda,\rho} q^{|\rho|} s_{\lambda/\rho}(x) s_{\lambda/\rho}(y) = \frac{1}{(q;q)_\infty} \prod_{i,j} \frac{1}{(x_i y_j; q)_\infty}$$
- $$\sum_{\mu} b_\mu \mathscr{P}_\mu(x) \mathscr{P}_\mu(y) = \prod_{i,j} \frac{1}{(x_i y_j; q)_\infty}$$

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$$(\star) \Leftrightarrow \sum_{\substack{\lambda,\rho \\ \lambda_1 = k}} q^{|\rho|} s_{\lambda/\rho}(x) s_{\lambda/\rho}(y) = \sum_{\mu_1+\nu_1=k} q^{|\nu|} b_\mu \mathscr{P}_\mu(x) \mathscr{P}_\mu(y)$$

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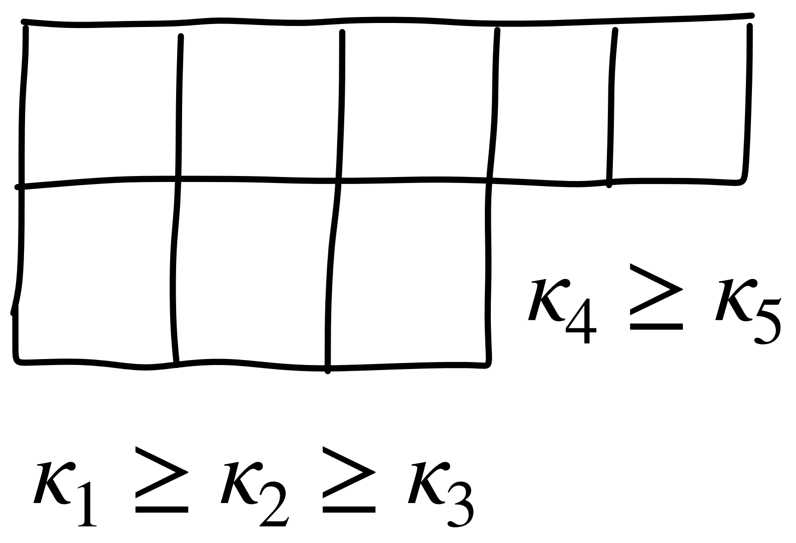
- $\mathscr{K}(\mu) = \{\kappa = (\kappa_1, \ldots, \kappa_{\mu_1}) : \kappa_i \geq \kappa_{i+1}$ if $\mu'_i = \mu'_{i+1}\}$

$$b_\mu = \prod_{i \geq 1} (q;q)^{-1}_{\mu_i - \mu_{i+1}} = \sum_{\kappa \in \mathscr{K}(\mu)} q^{\kappa_1 + \kappa_2 + \cdots + \kappa_{\mu_1}}$$

$\mu =$ 

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IDEA: $(P, Q) \overset{\Upsilon}{\longleftrightarrow} (V, W; \kappa, \nu)$

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Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)

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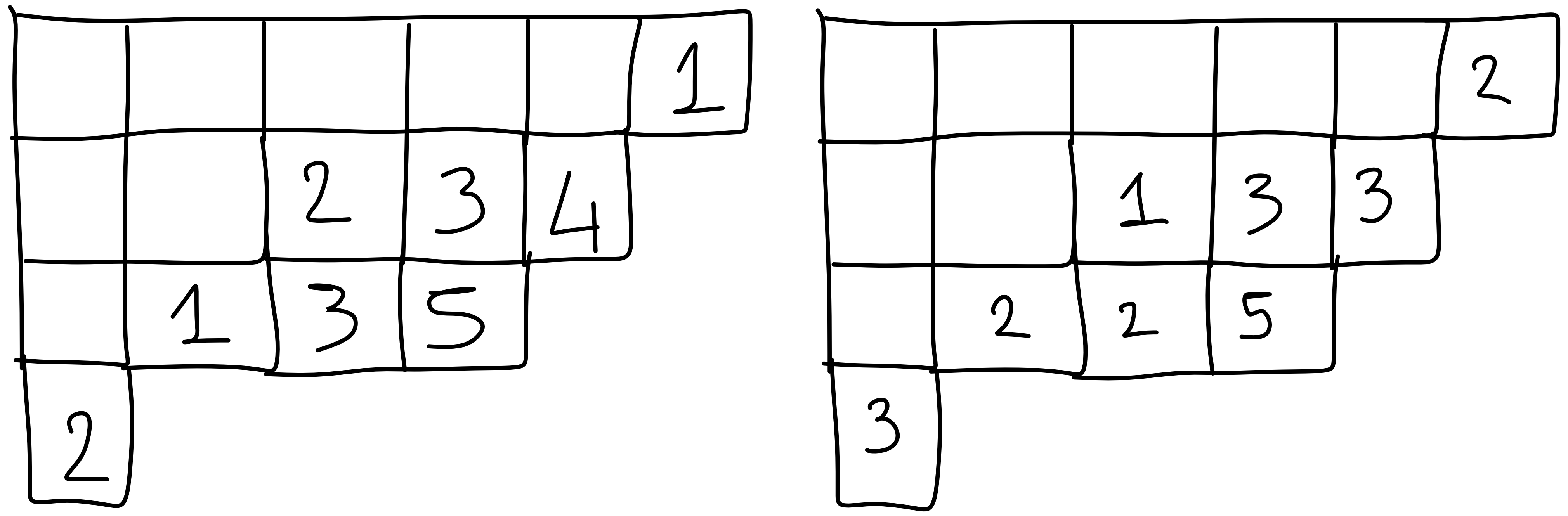
P $\qquad$ Q



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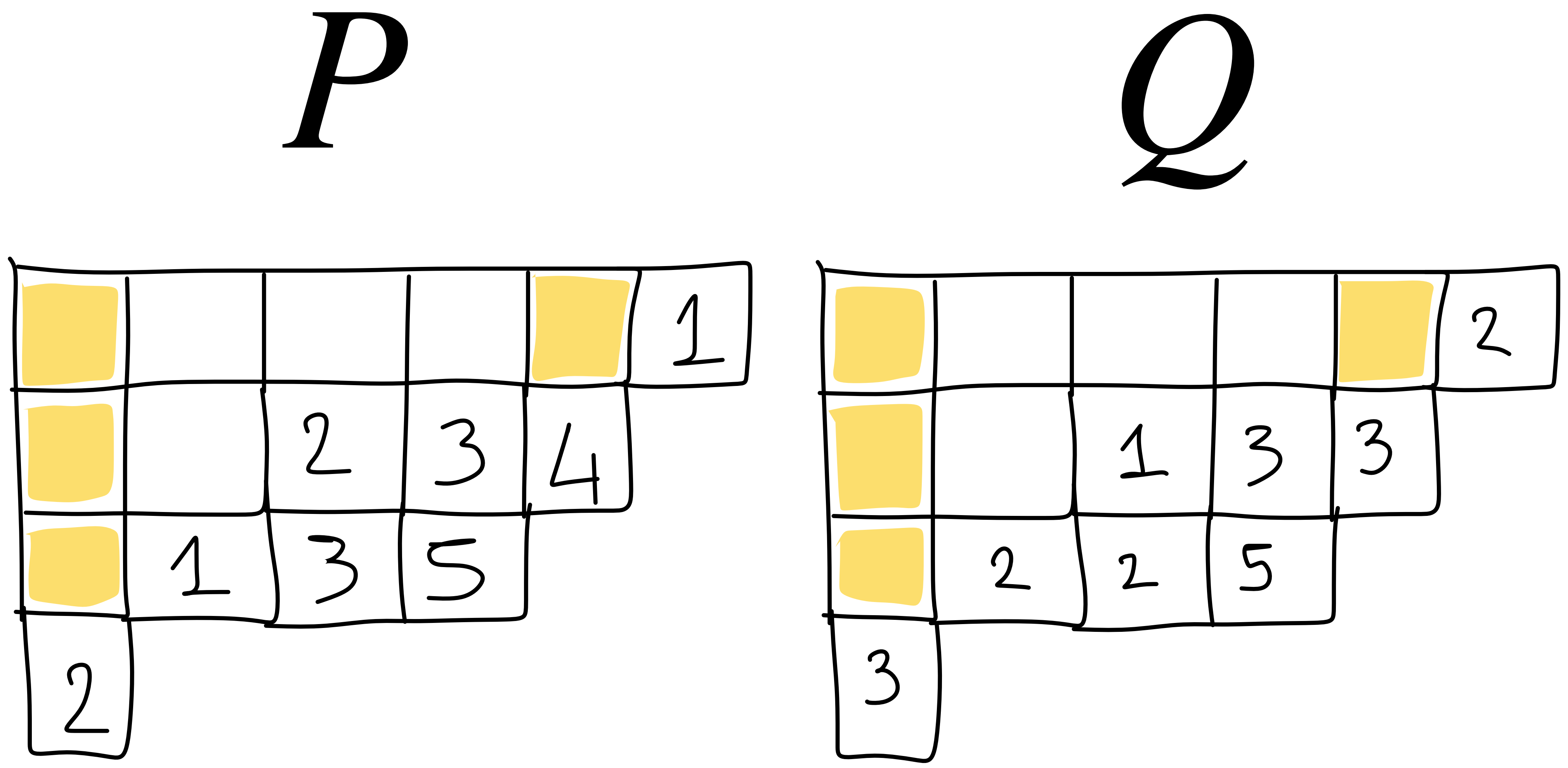

- First construct ν by "squeezing" P, Q

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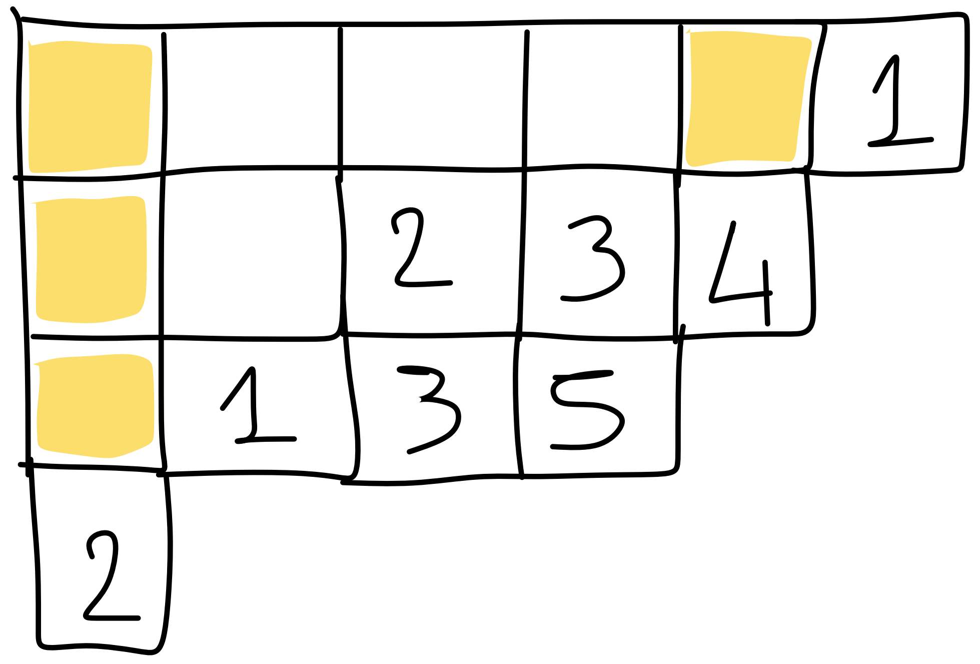
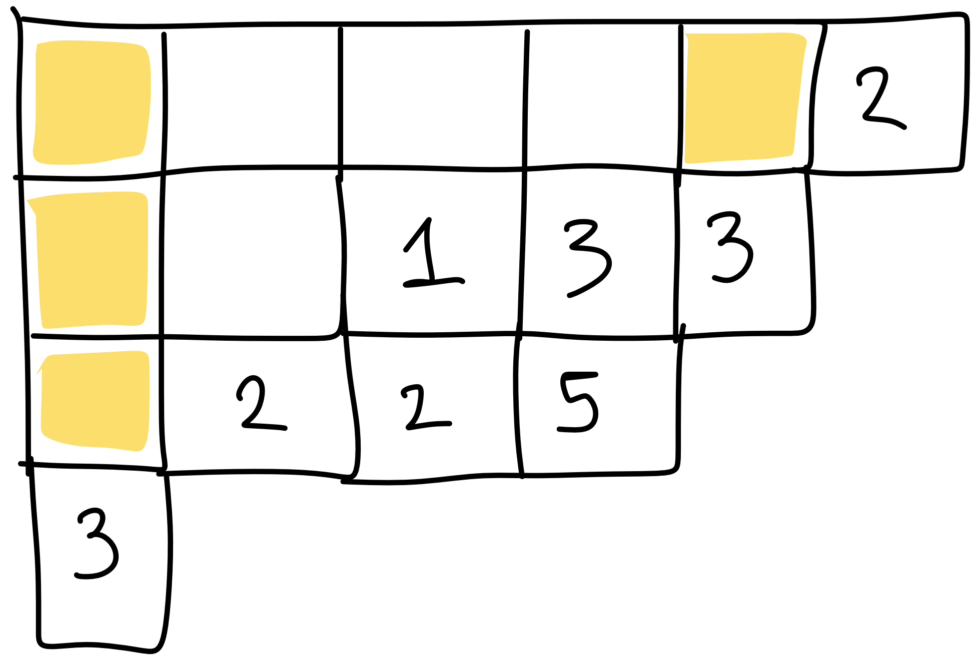
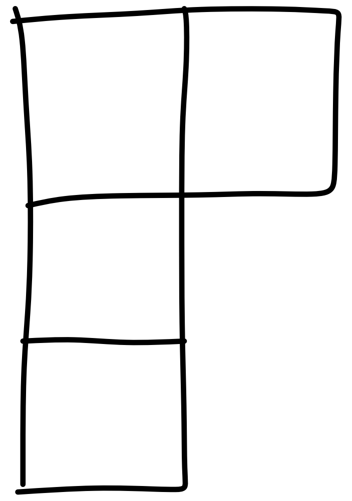
P



Q



$\nu =$ 

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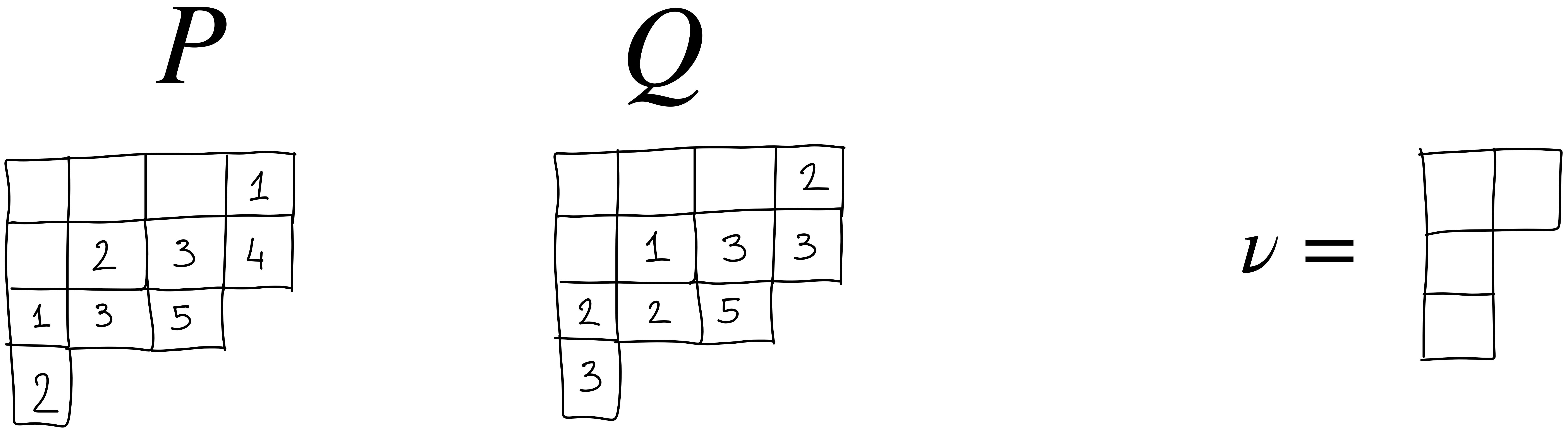
Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)



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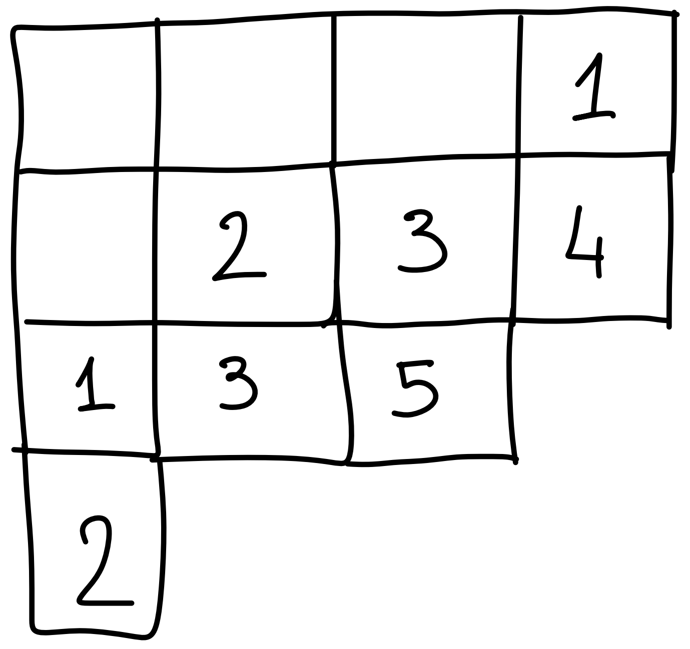
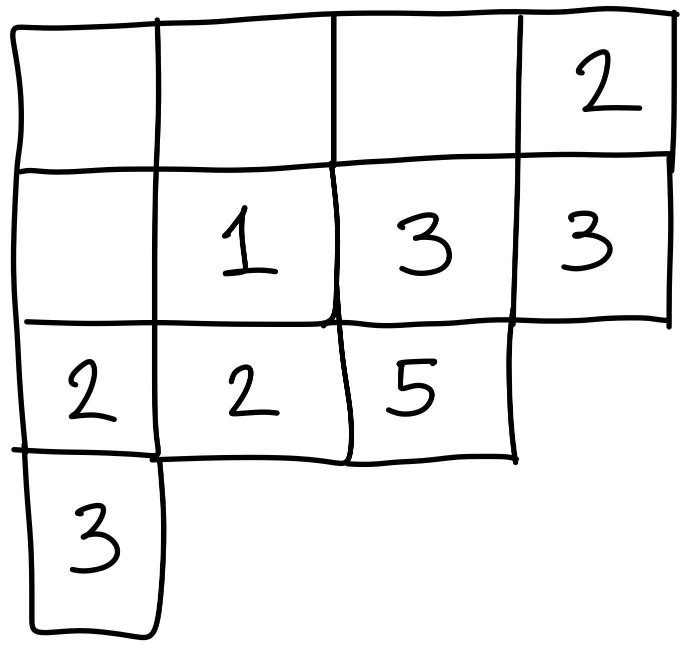
✔

P



Q



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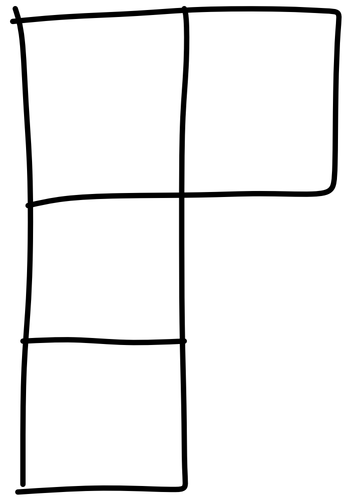


- First construct ν by “squeezing” P, Q
- From now on assume pair (P, Q) is “squeezed”

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)

P

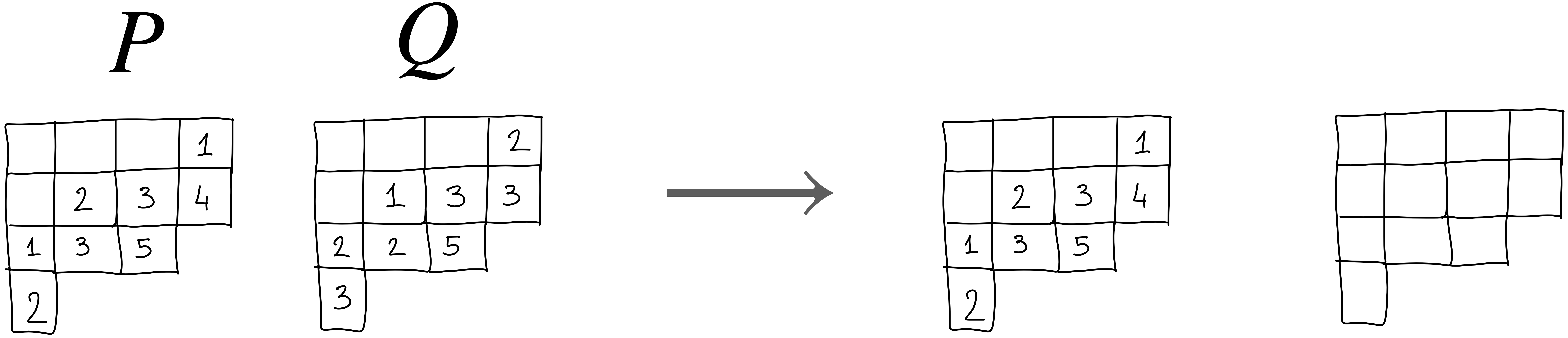
			1
	2	3	4
1	3	5	
2			

Q

			2
	1	3	3
2	2	5	
3			

- To construct (V, W) we define a dynamics on tableaux based on internal insertion [Sagan-Stanley'90]

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)



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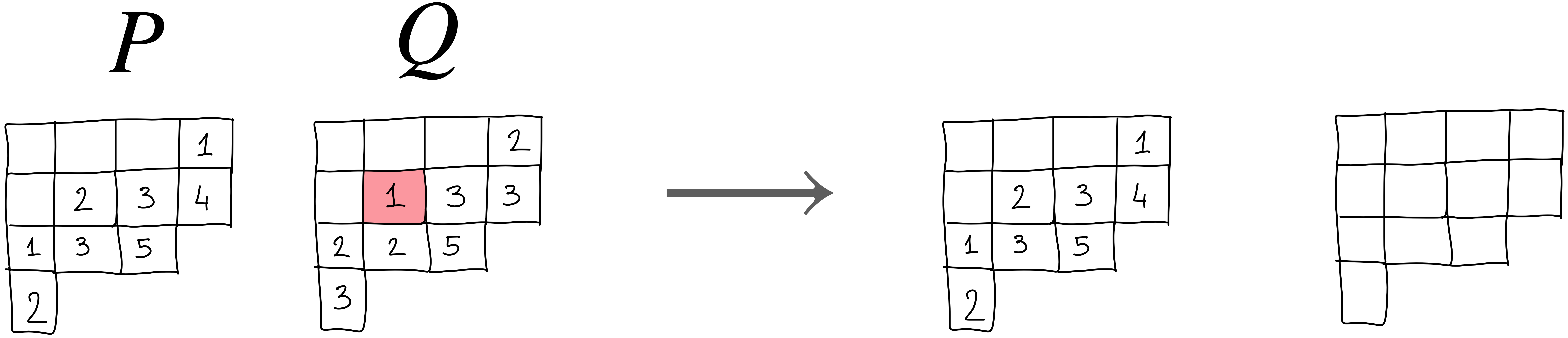

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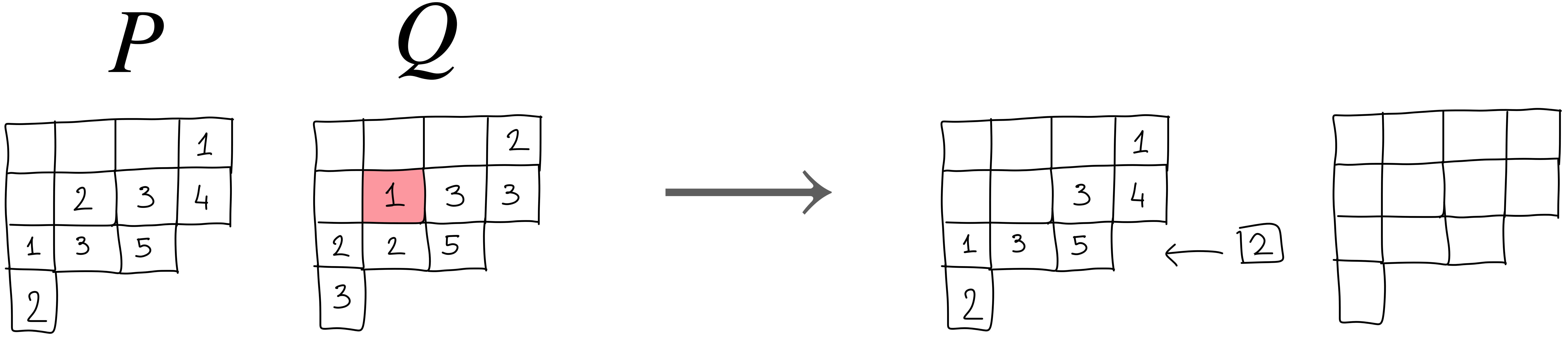

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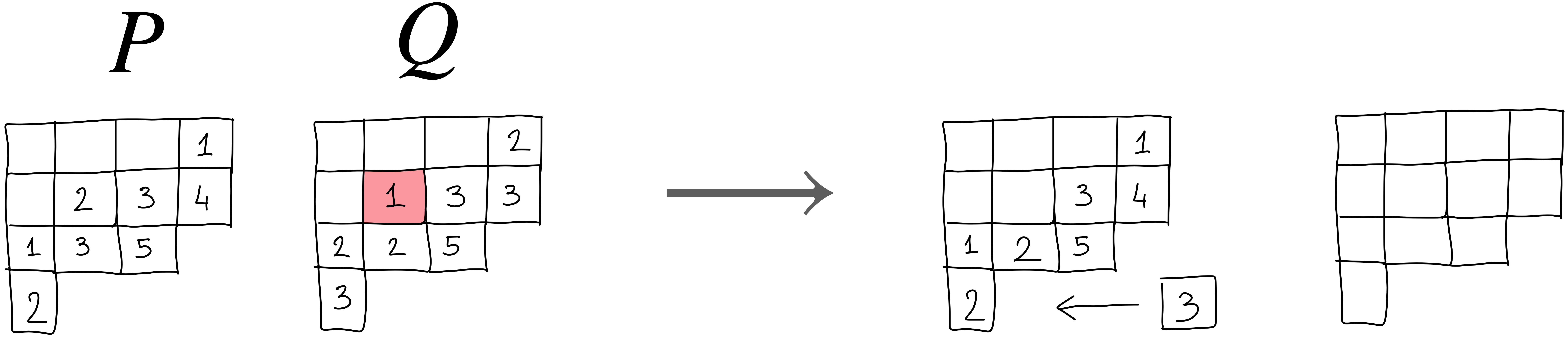

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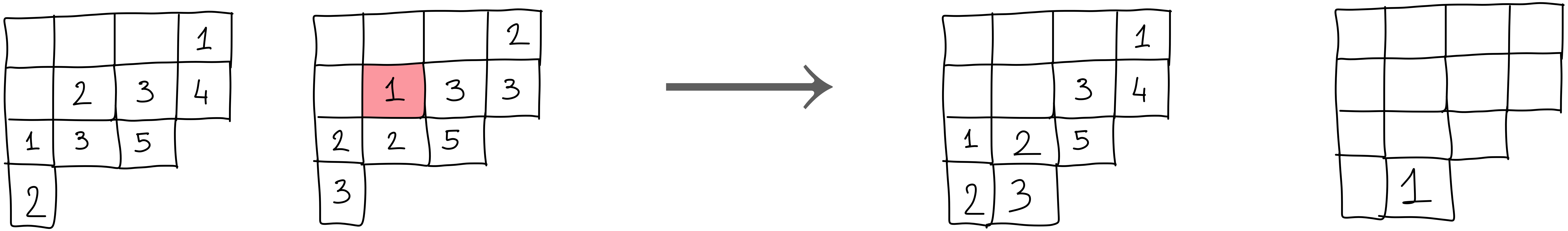

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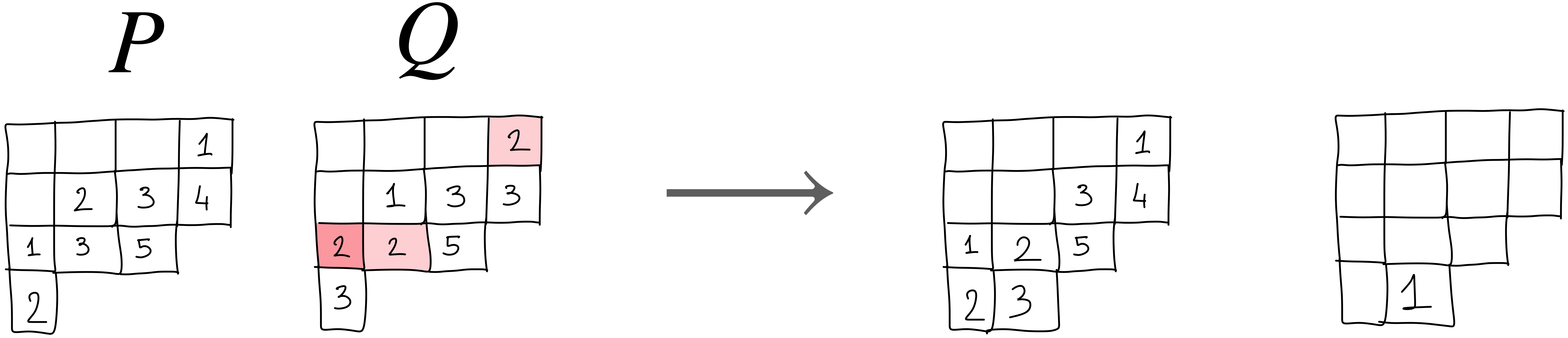
Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)



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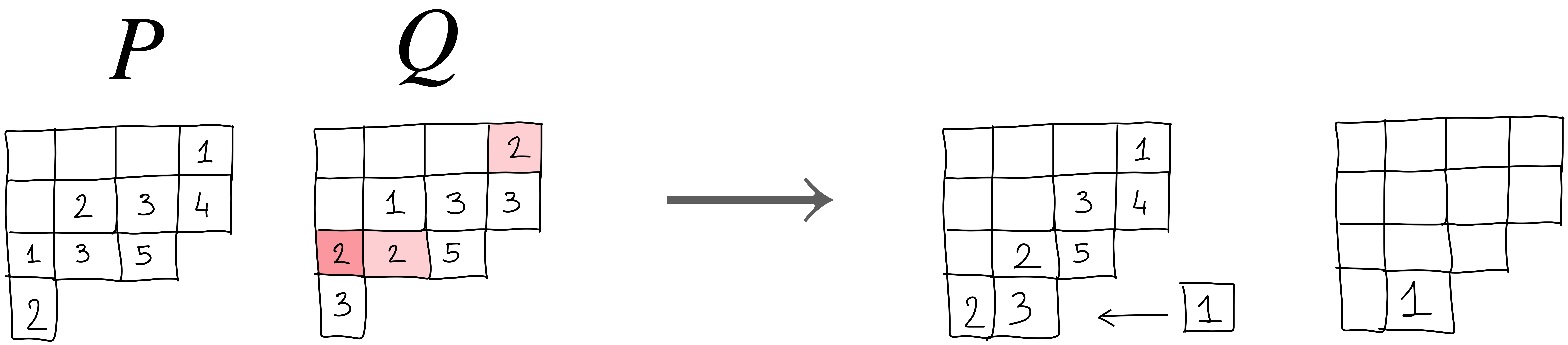

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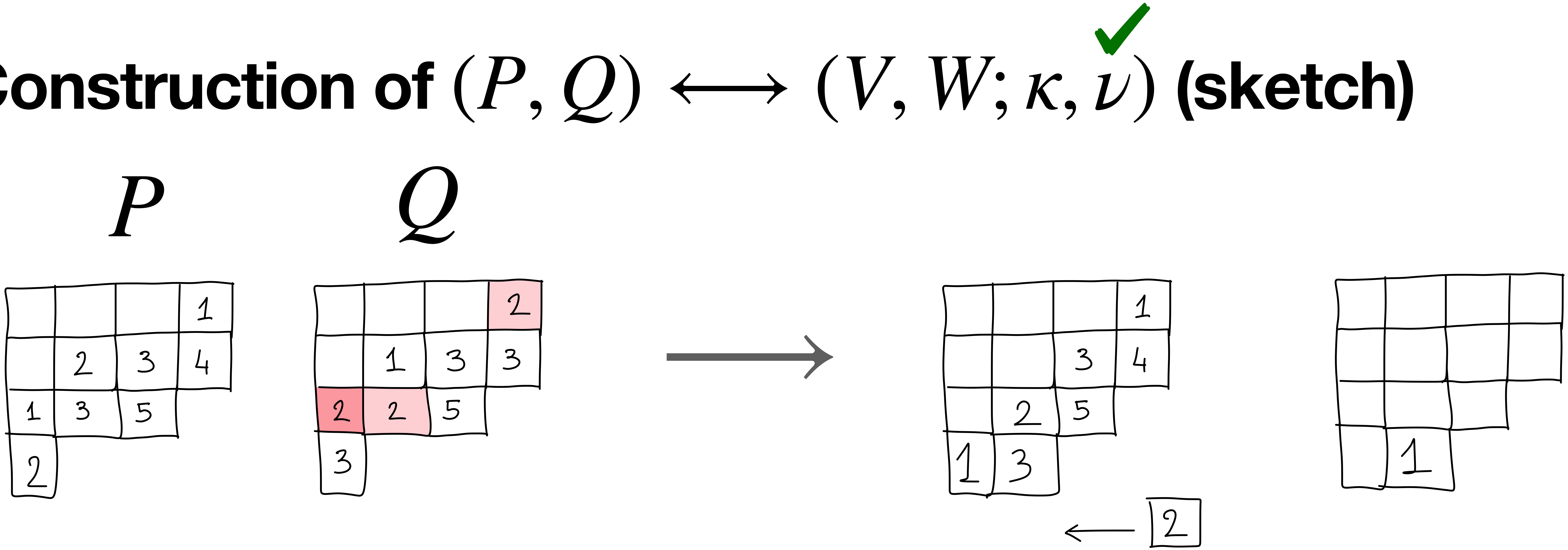

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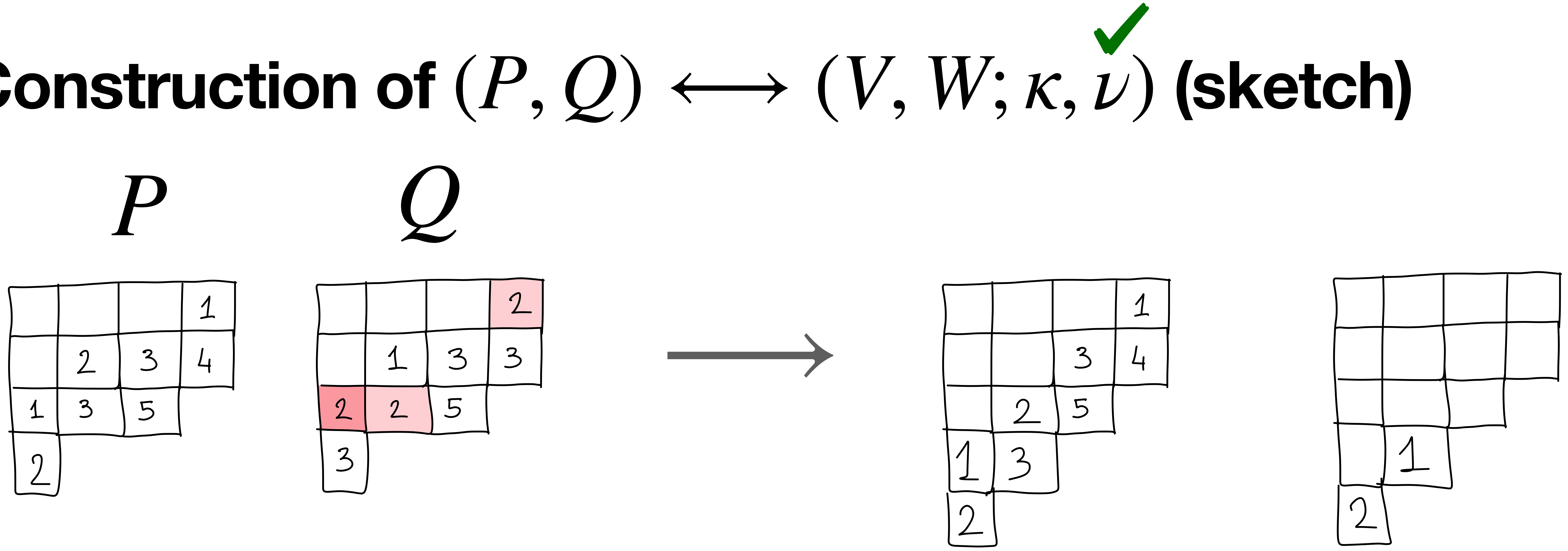

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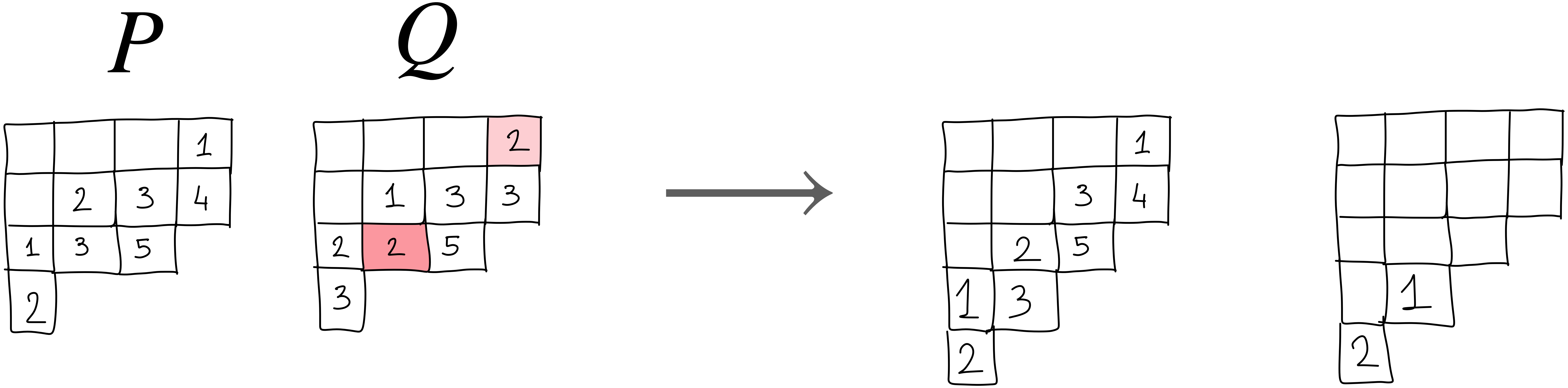

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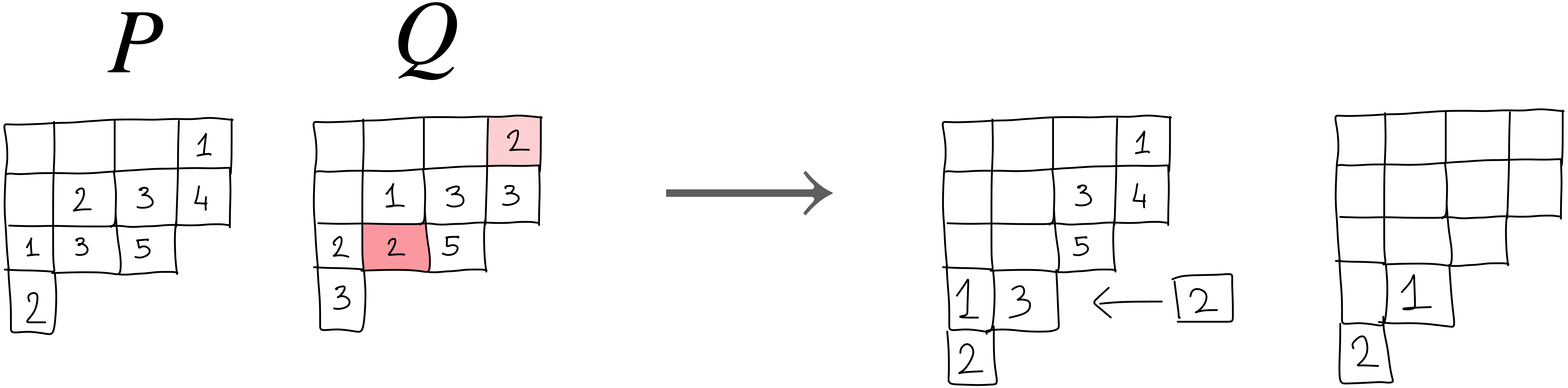

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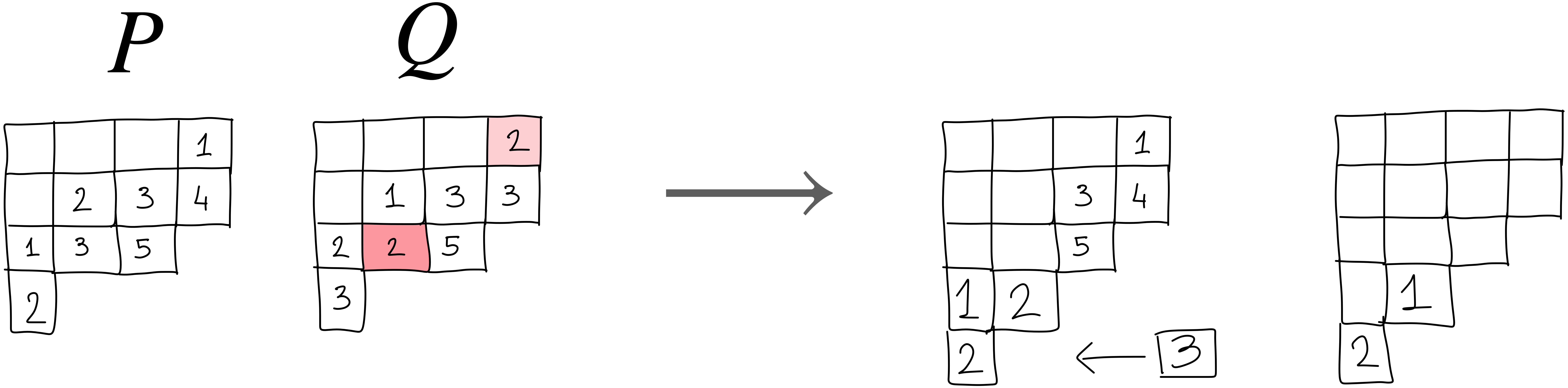

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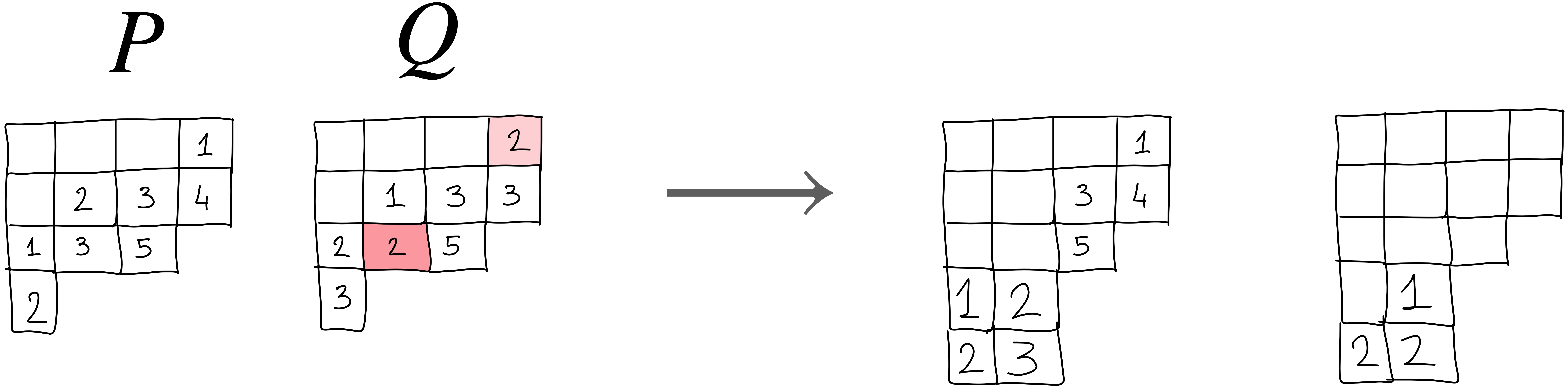

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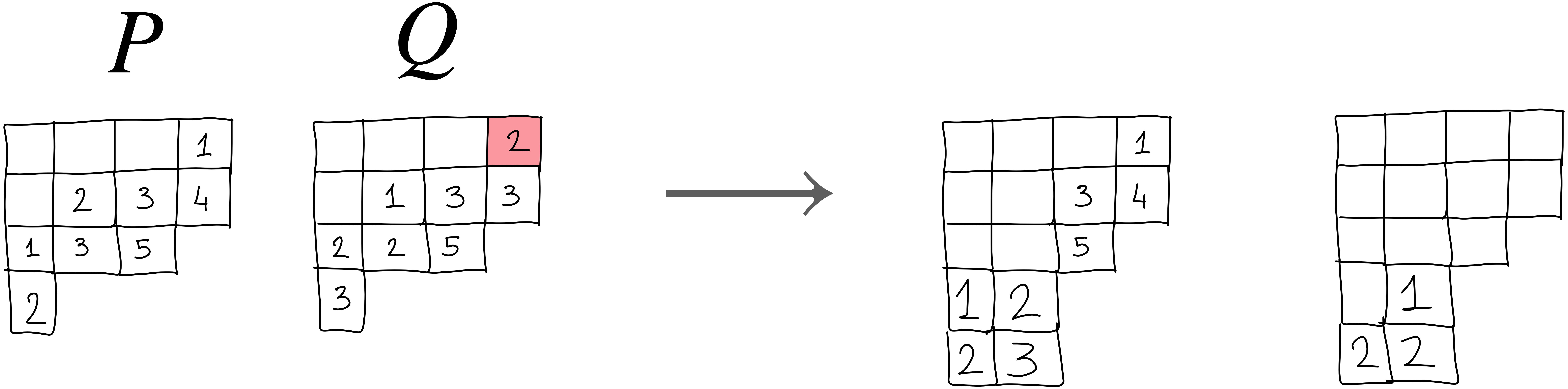

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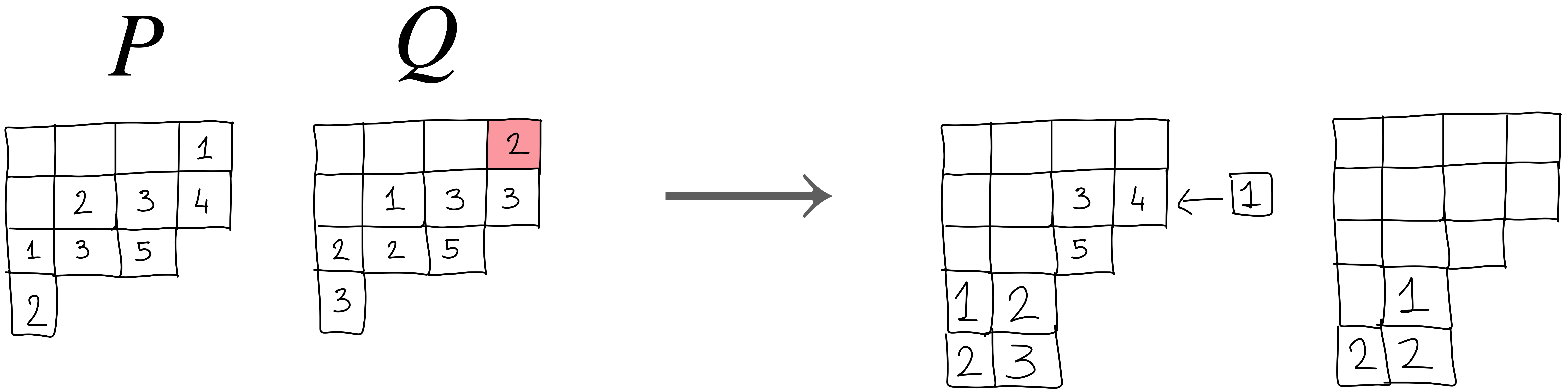

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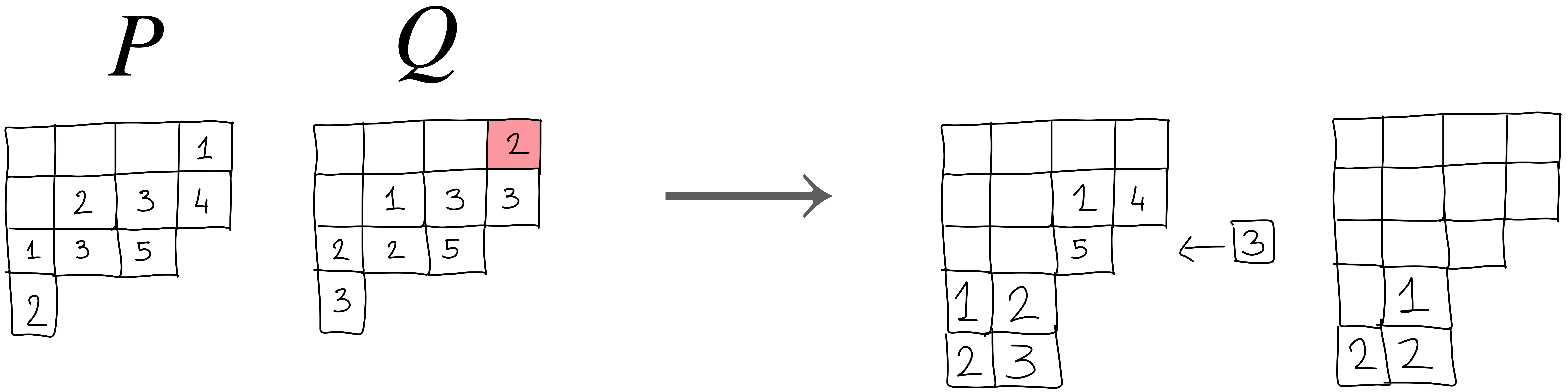

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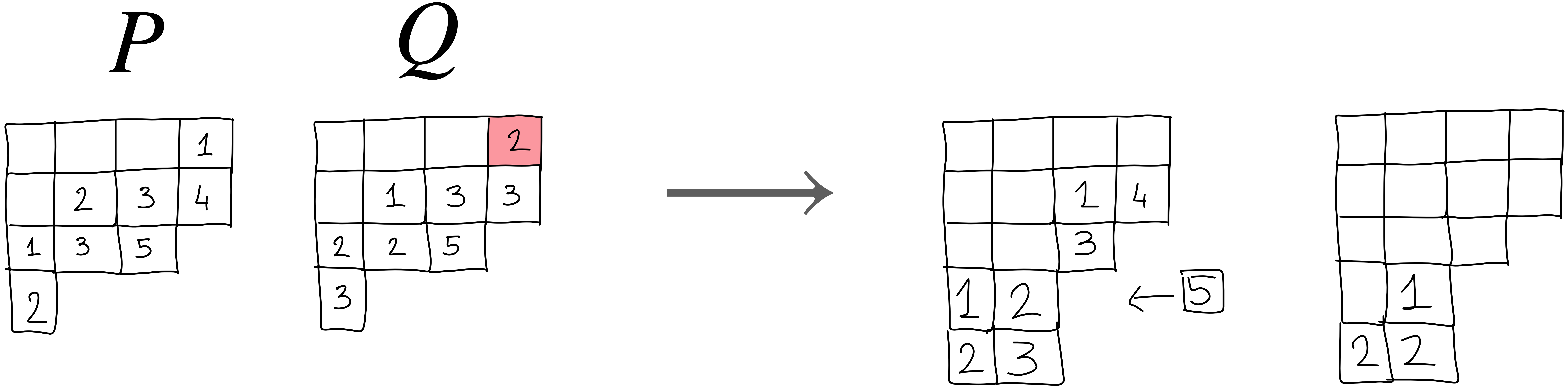

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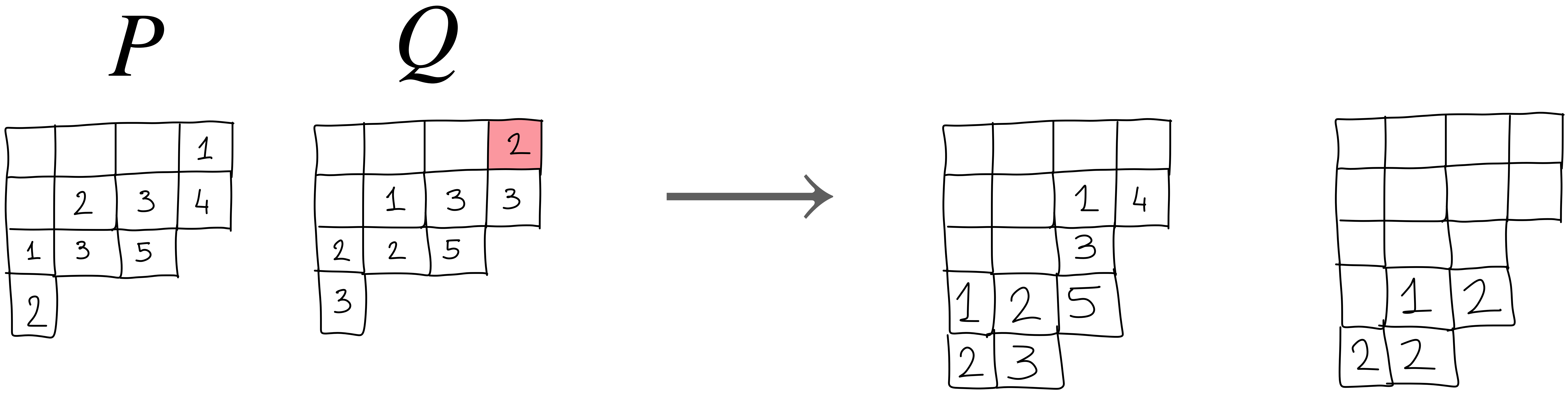

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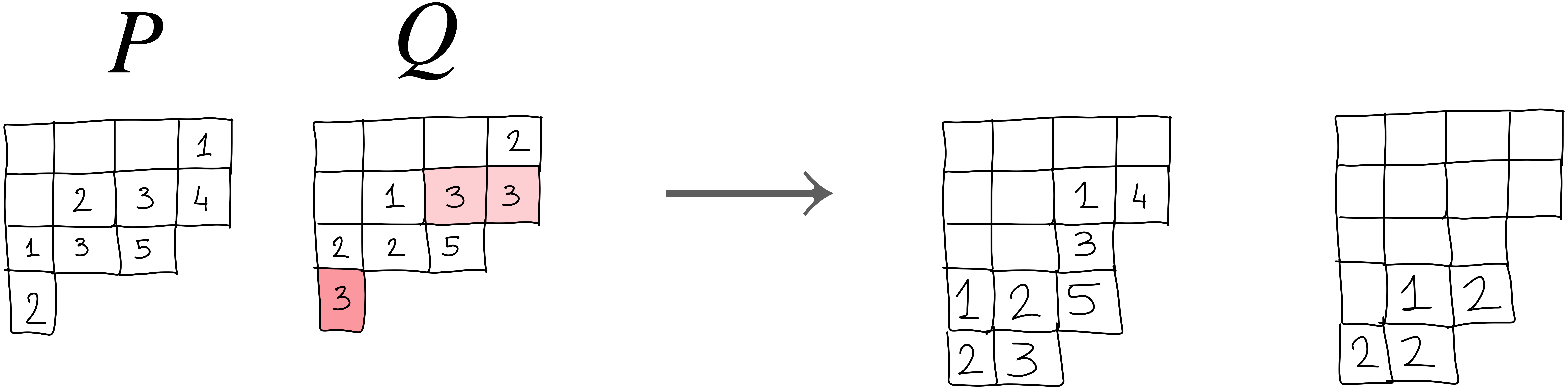

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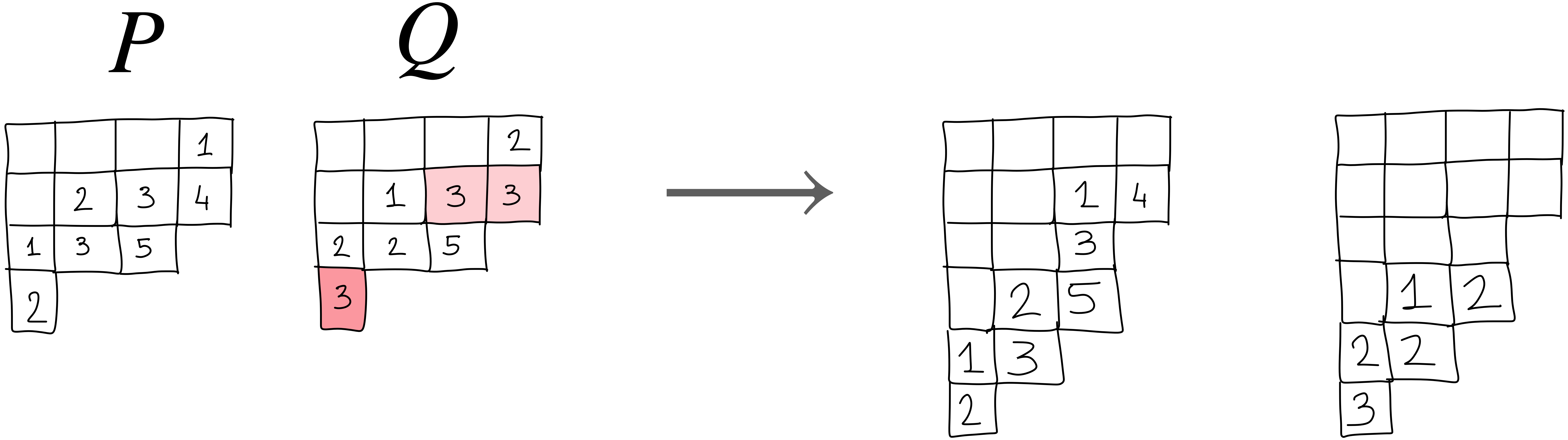

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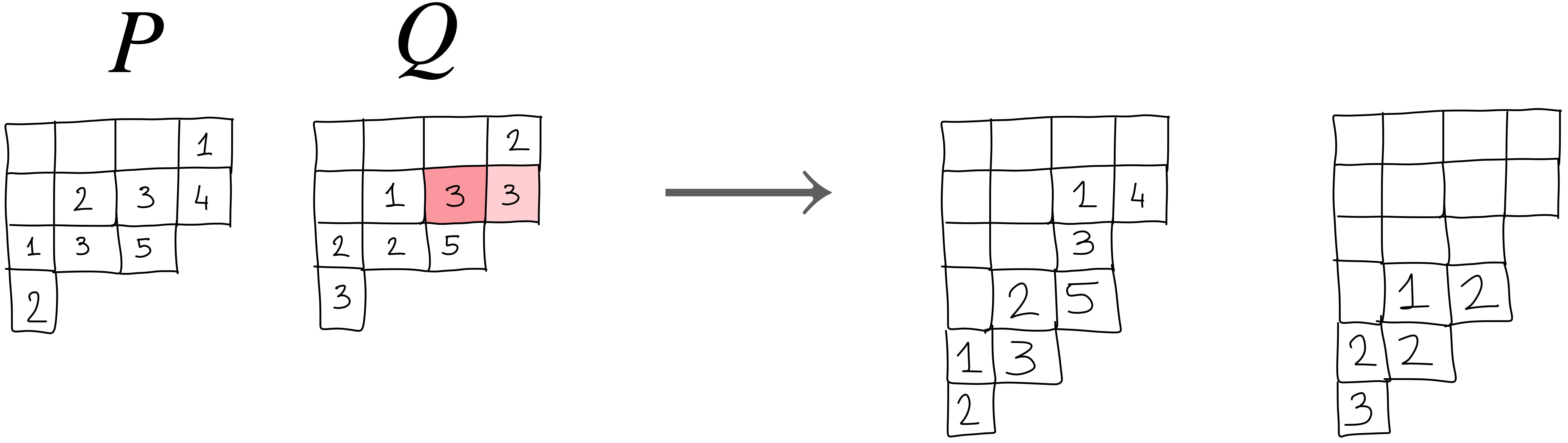

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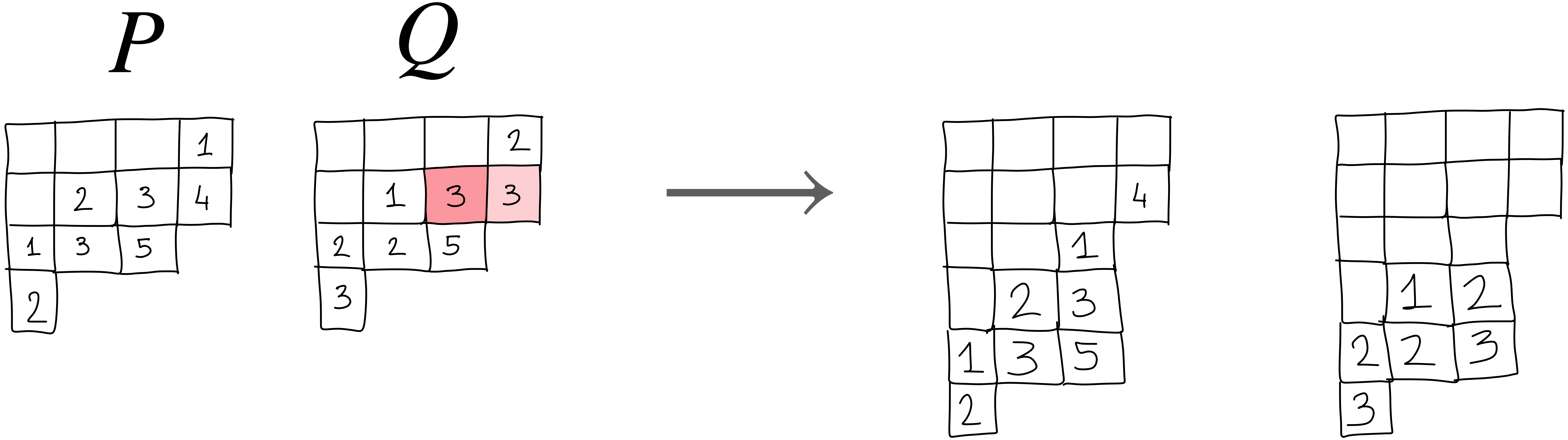

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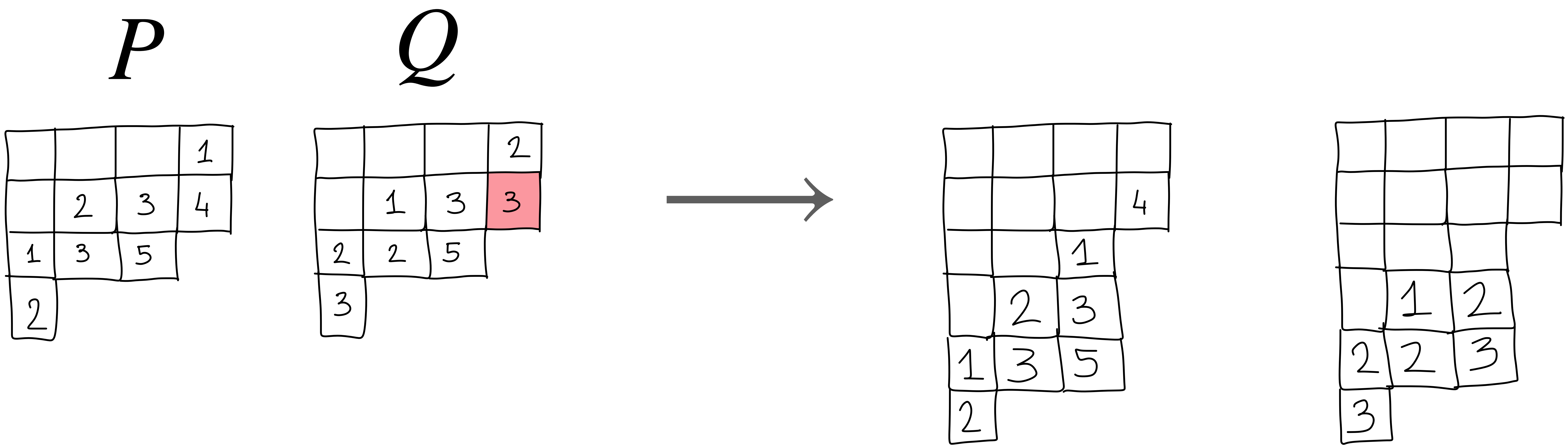

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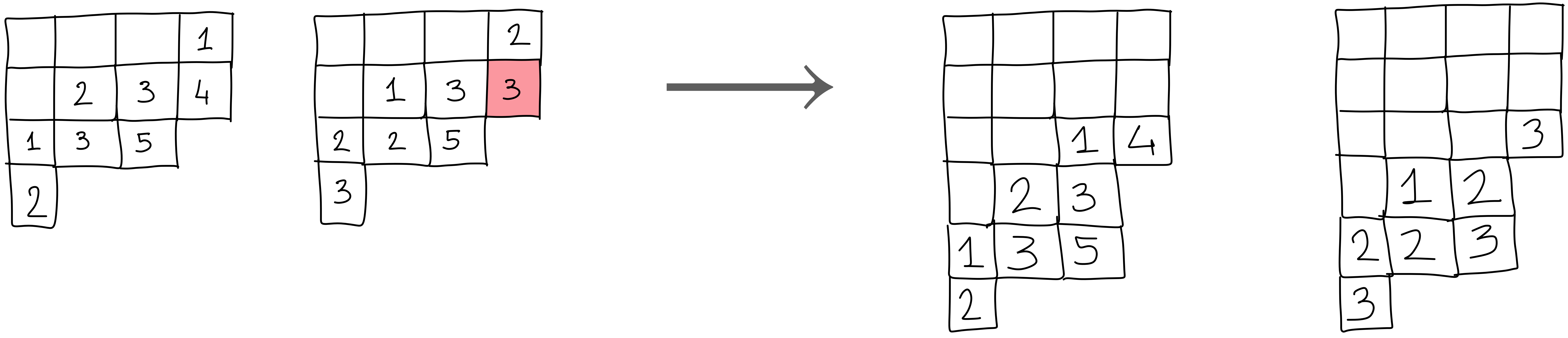

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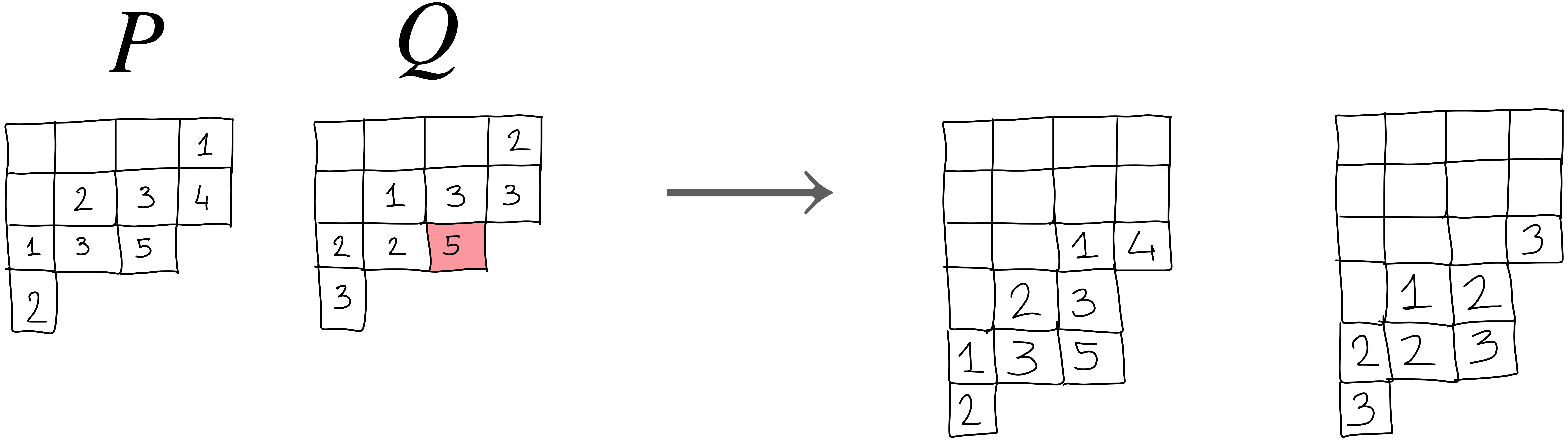

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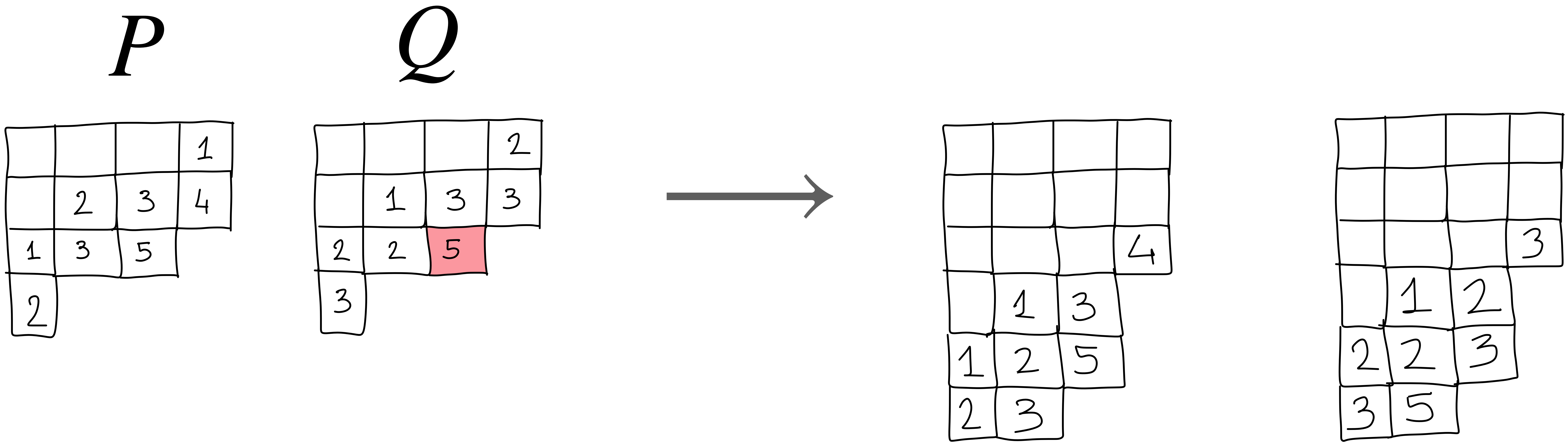

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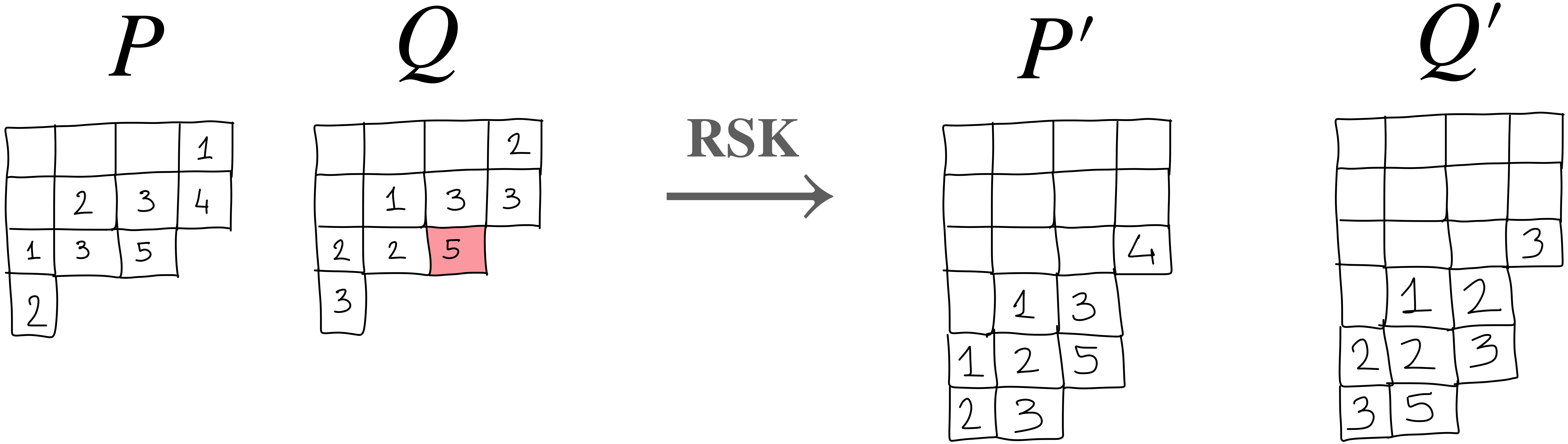

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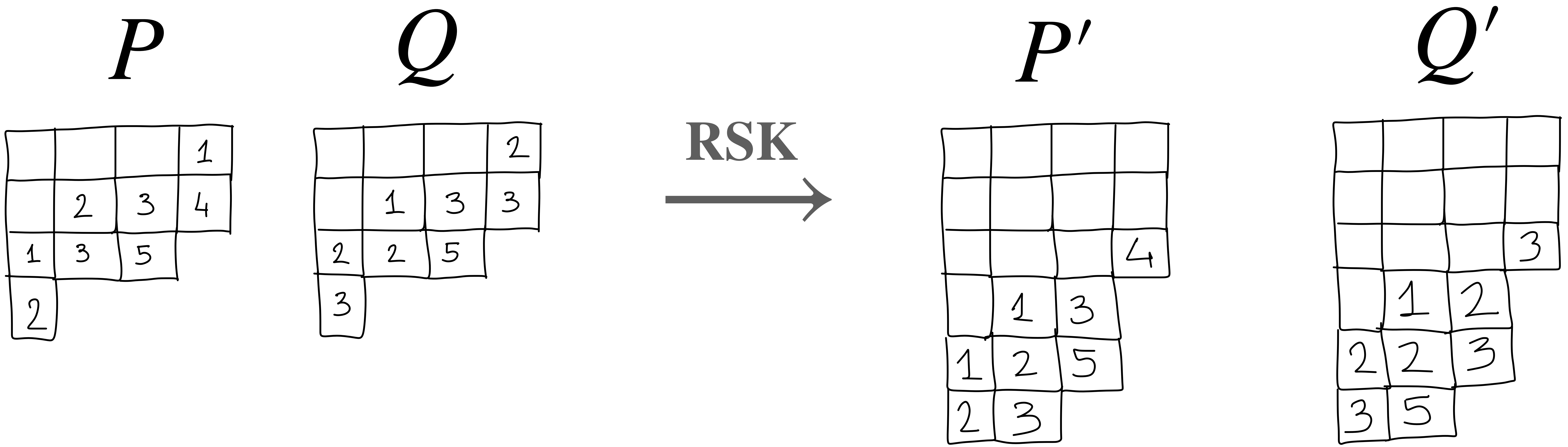

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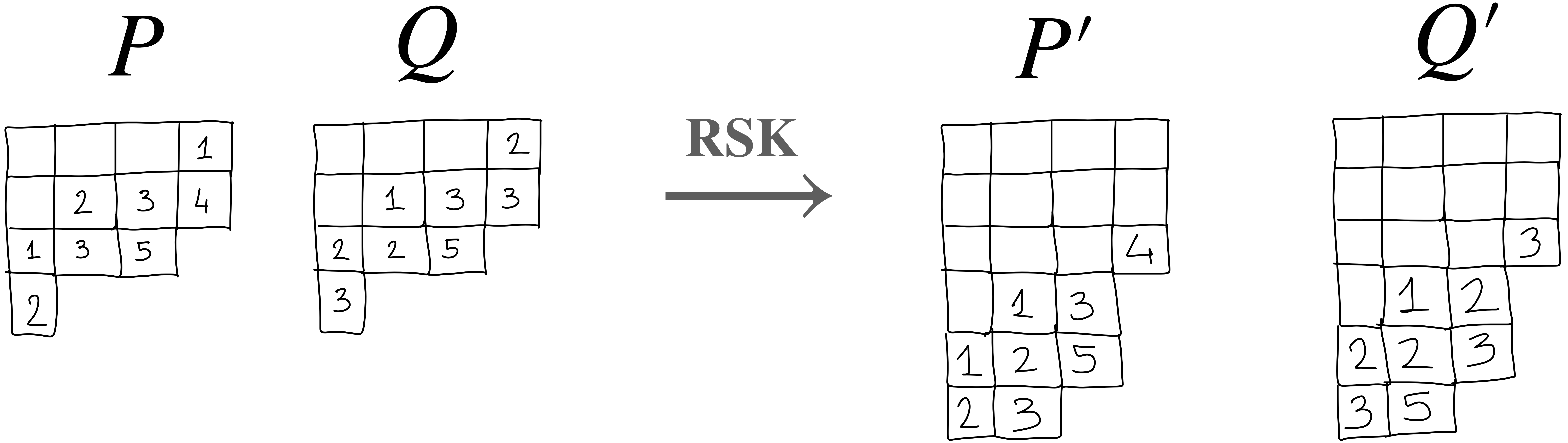
$$(P, Q) \to (P', Q')$$

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)



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$$(P, Q) \to (P', Q') \to \cdots \to (P^{(n)}, Q^{(n)})$$

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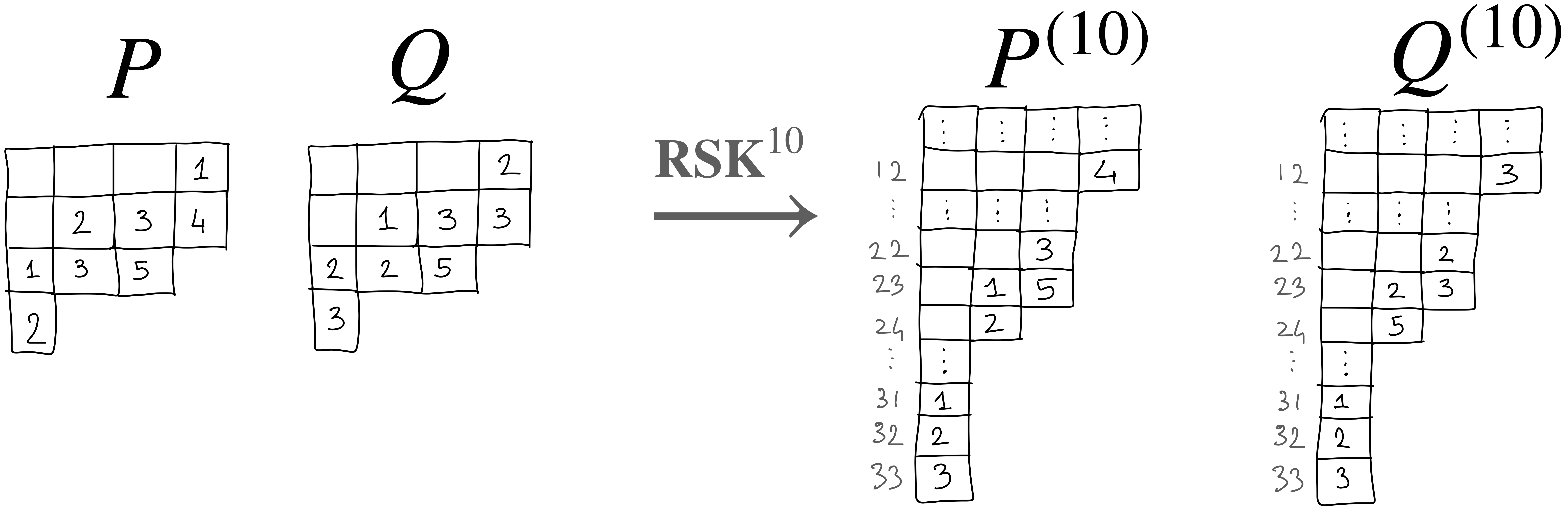

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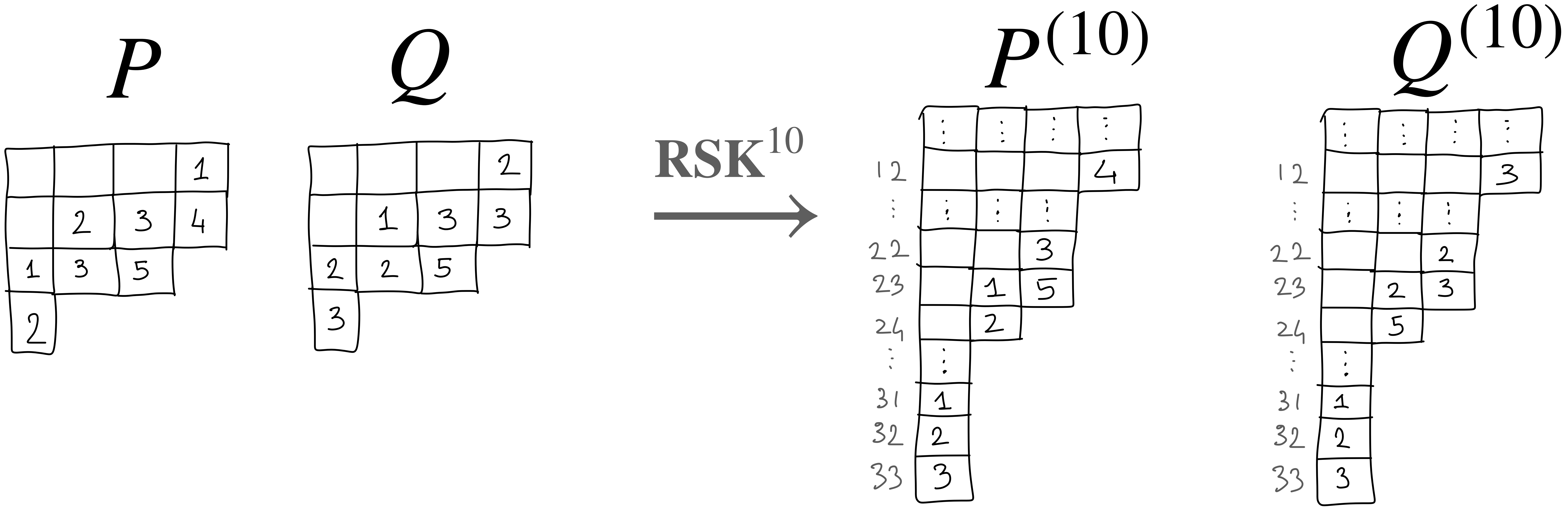

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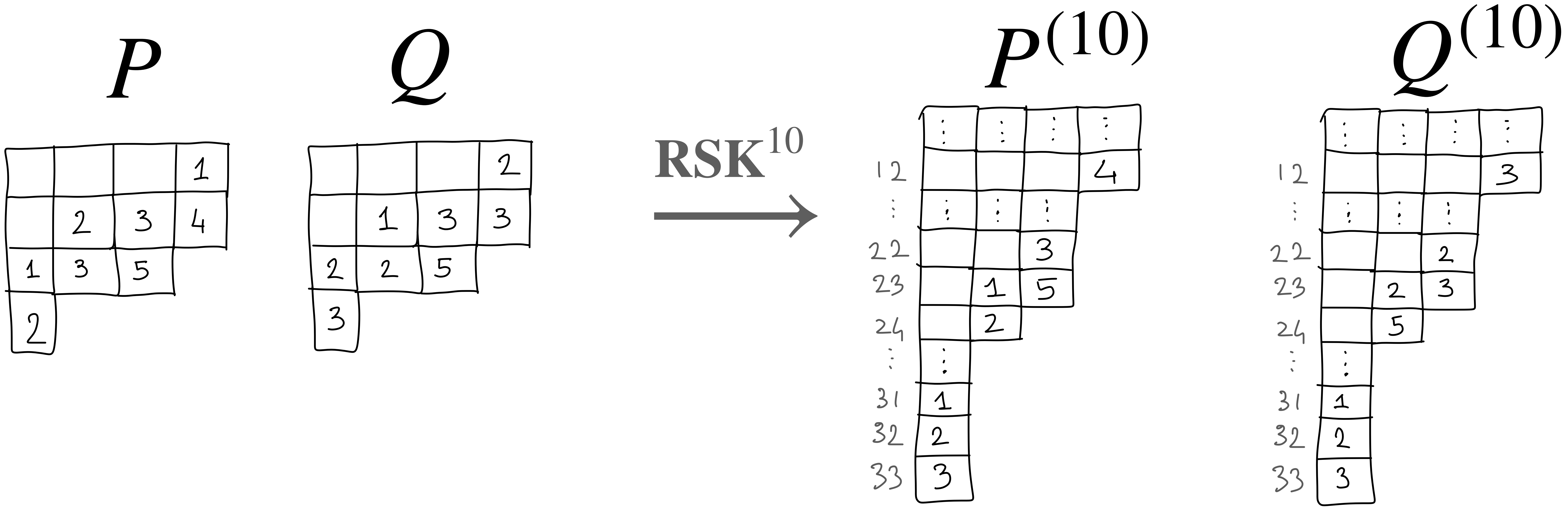
Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)



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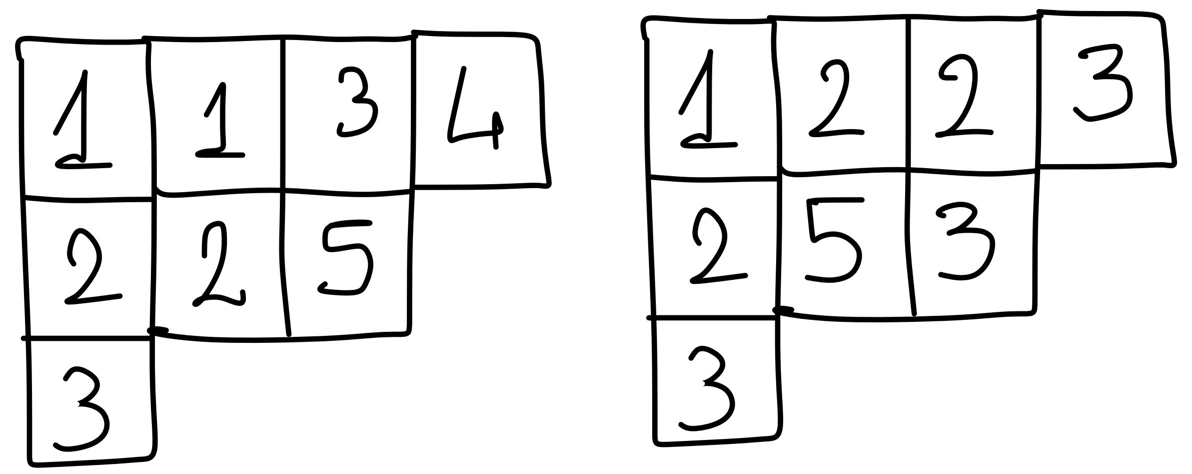

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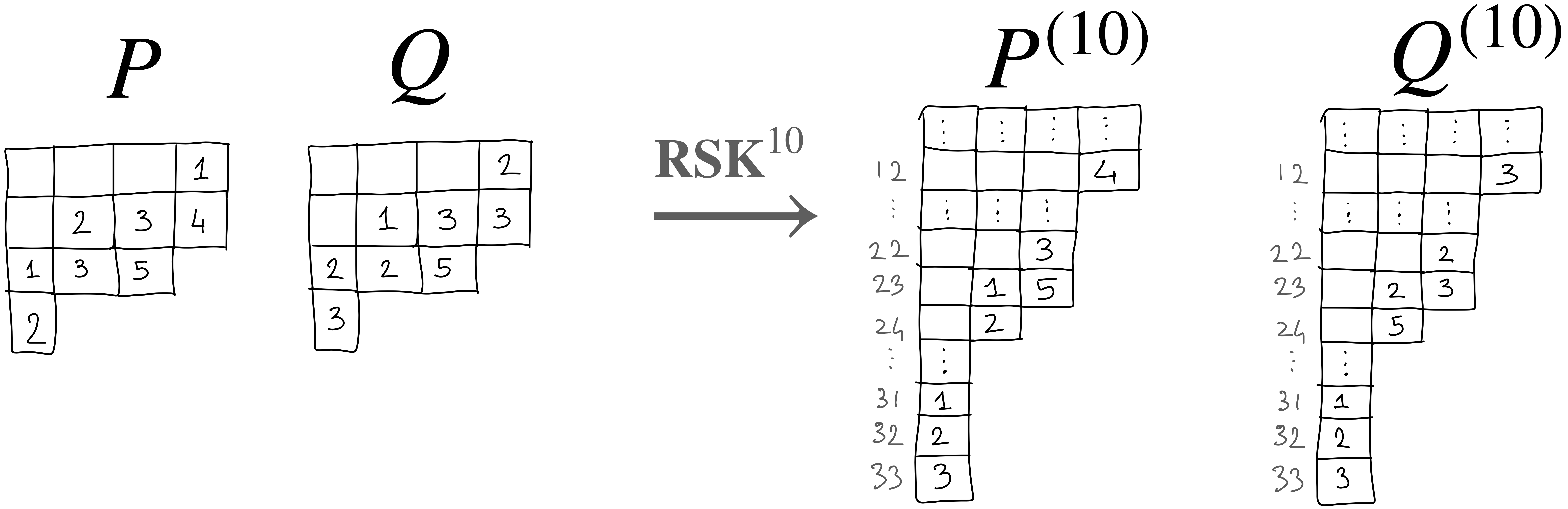

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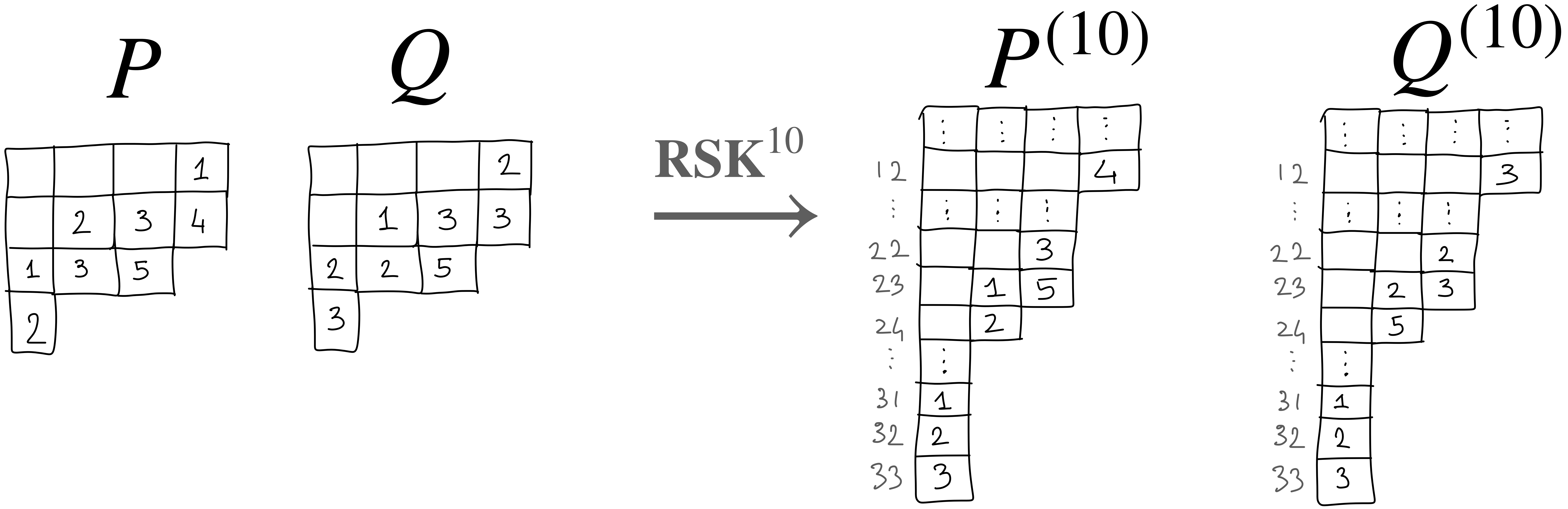
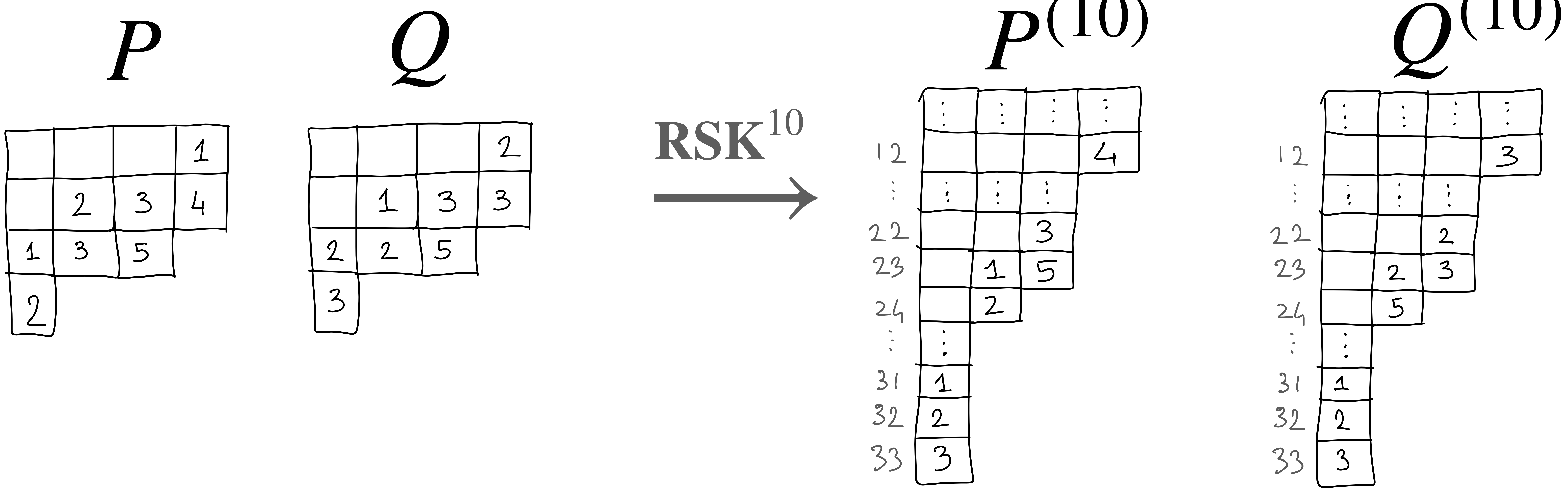
Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)



- It remains to construct κ.
- μ = shape of (V, W) $\quad \tilde{\lambda}/\tilde{\rho}$ = shape of $(P^{(n)}, Q^{(n)})$ for n large

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)



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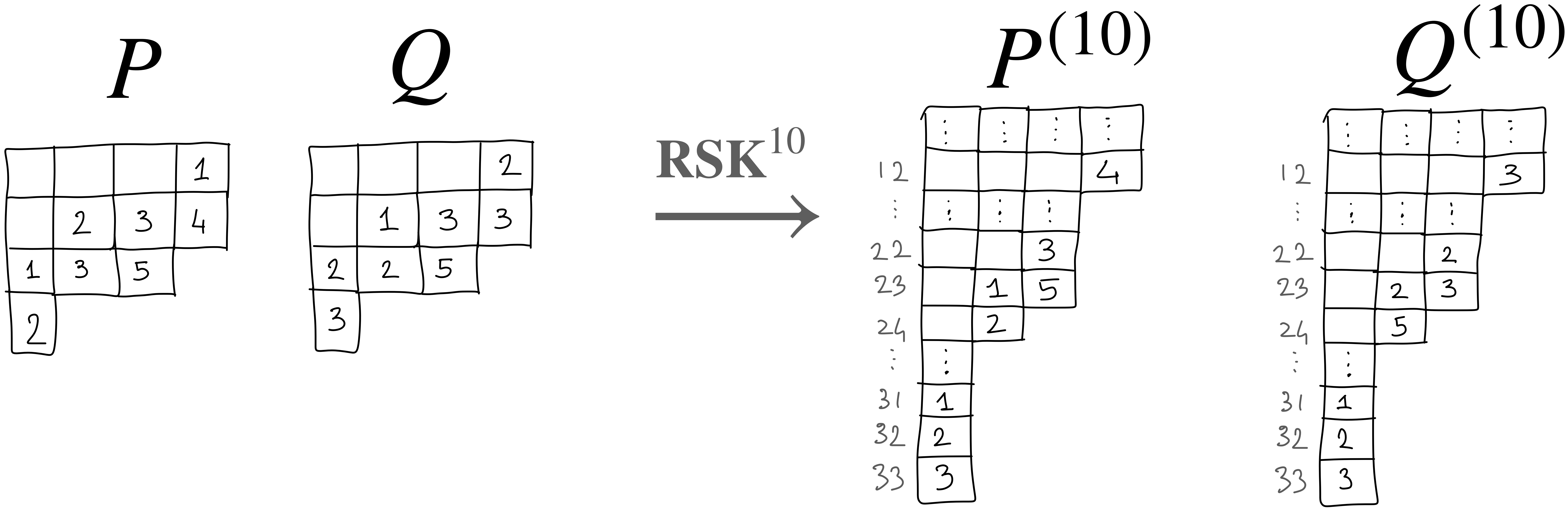

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$\tau_1 = 0 = 30 - 10 \times 3$

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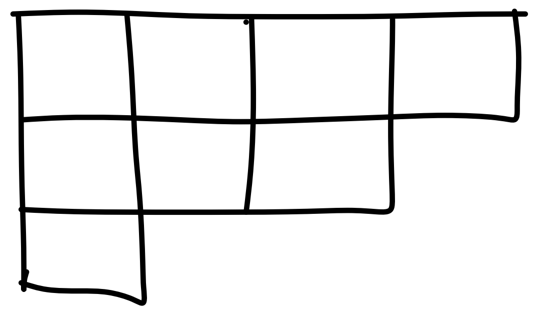
$\mu =$ 

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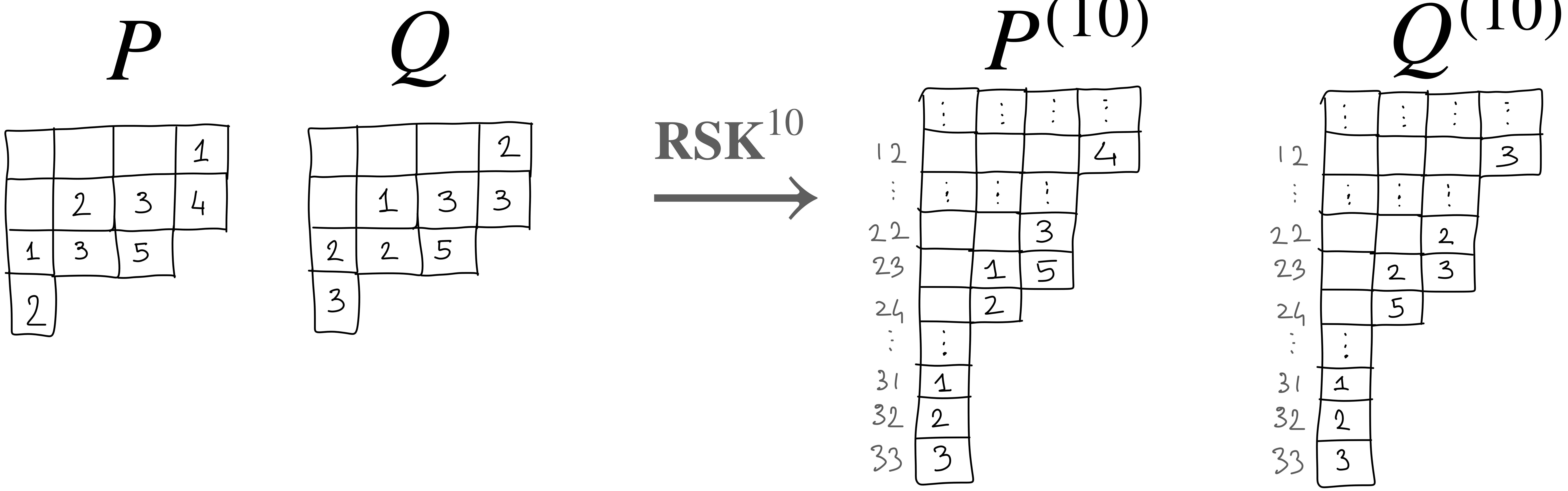

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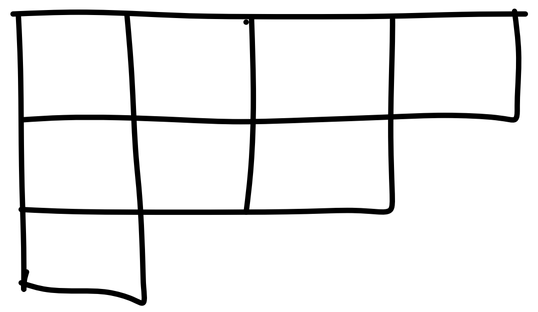
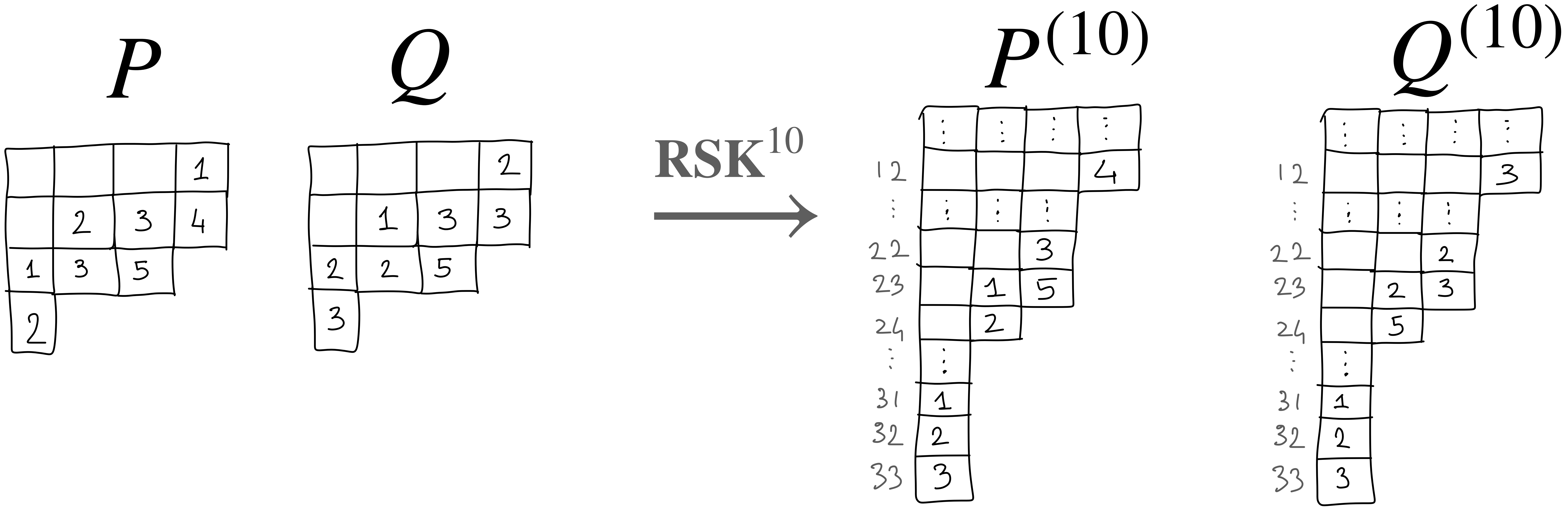
$\mu =$ 

$\tau_1 = 0 = 30 - 10 \times 3$

$\tau_2 = 2 = 22 - 10 \times 2$

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)



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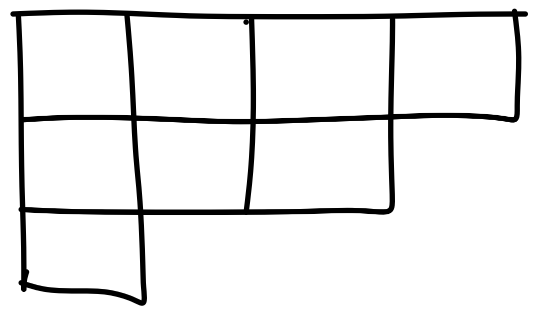
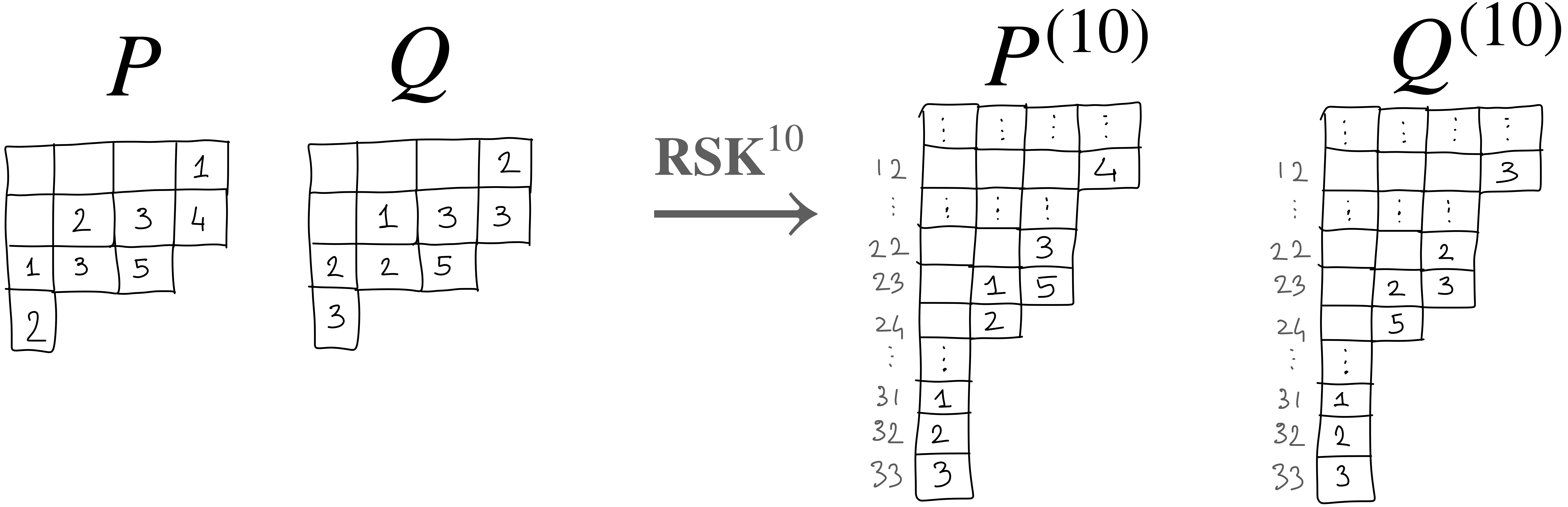
$\mu =$ 

$$\tau_1 = 0 = 30 - 10 \times 3$$
$$\tau_2 = 2 = 22 - 10 \times 2$$
$$\tau_3 = 1 = 21 - 10 \times 2$$

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)



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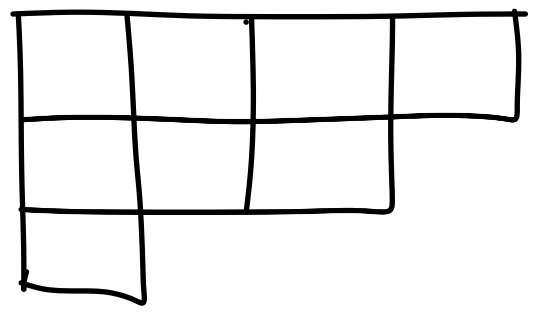
$\mu =$ 

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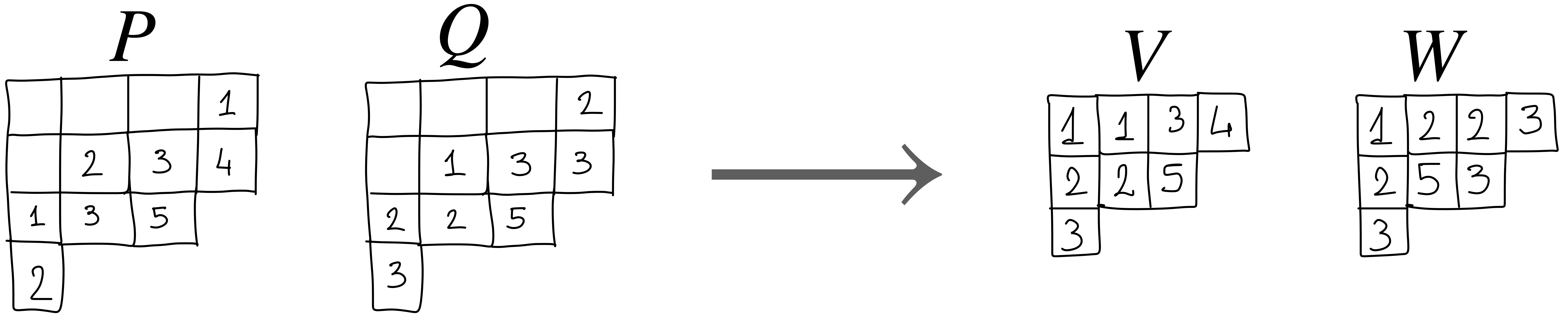
$\tau_4 = 1 = 11 - 10 \times 1$

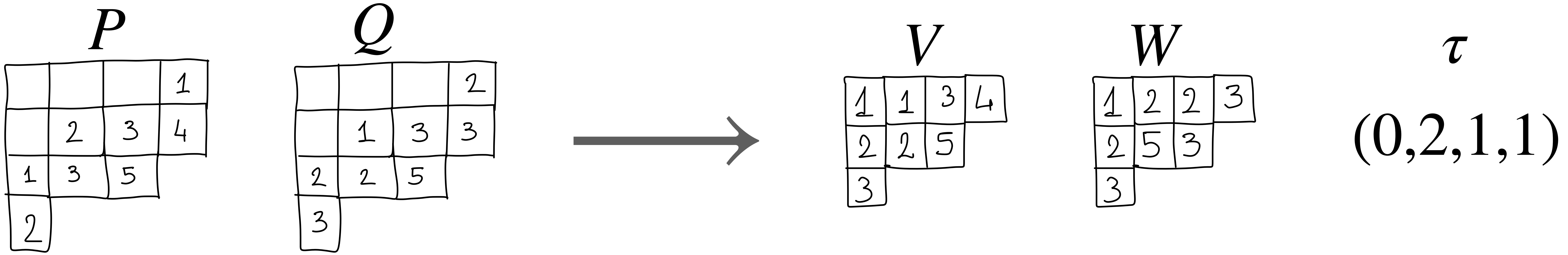
Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)



τ

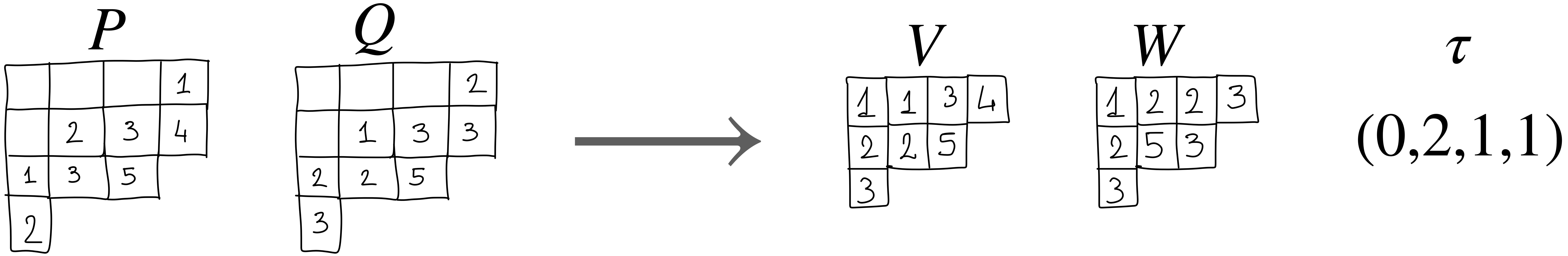
(0,2,1,1)

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)



- $(P, Q) \leftrightarrow (V, W; \tau)$ is a bijection, but τ depends on (V, W)!

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)



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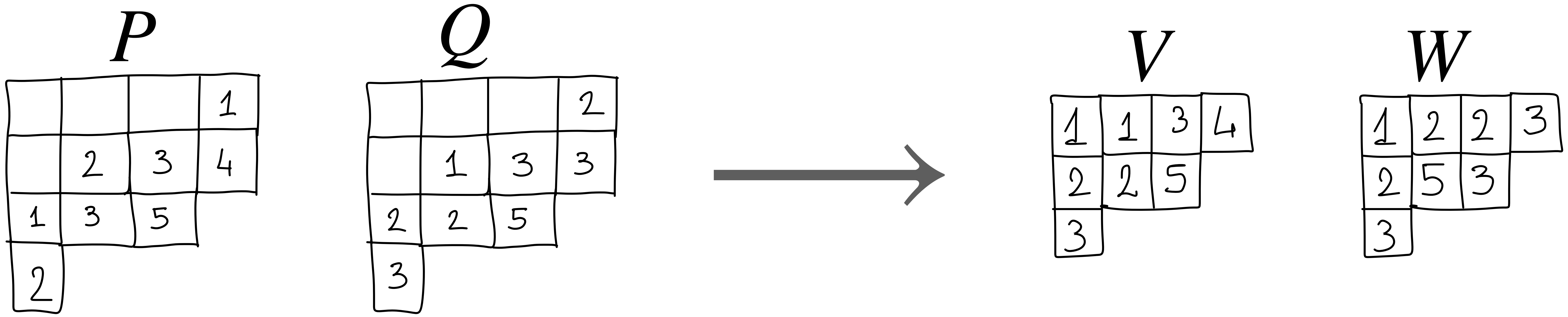
LEMMA (IMS'21)

$$\tau_i = \kappa_i + \mathscr{H}_i(V) + \mathscr{H}_i(W) \qquad (\triangle)$$

$\mathscr{H}_i$ = local energy function

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)



τ

$(0,2,1,1)$

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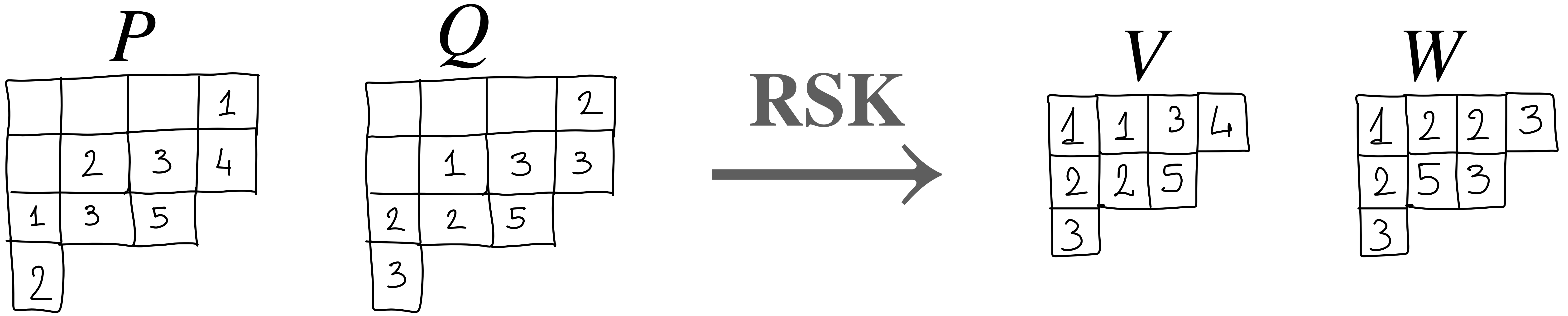
$\mathscr{H}_i$ = local energy function

Comments on ($\triangle$):

- Shows deep connections between theories of skew tableaux and theories of discrete integrable systems, Kashiwara crystals, …
- $\mathscr{H}_i$ appear in the description of *phase shift* in the $\widehat{\mathfrak{sl}}_n$ Box and Ball System
- This is because the skew RSK dynamics is a generalization of the Box and Ball Systems

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)



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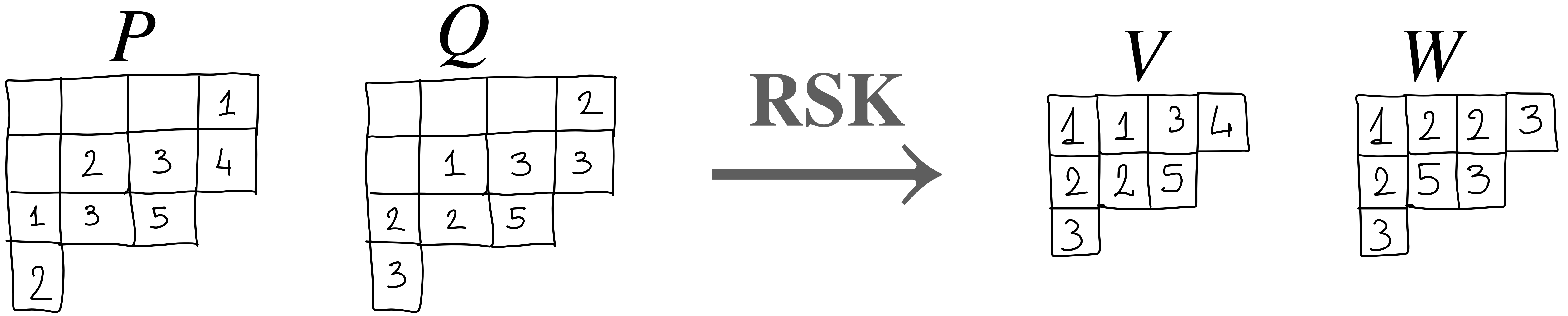
$(0,2,1,1)$

$$\tau_i = \kappa_i + \mathscr{H}_i(V) + \mathscr{H}_i(W)$$

- To prove energy ($\mathscr{H}$) formulas we need to study symmetries of the skew RSK
- Classical RSK commutes with $\mathfrak{sl}_n$ Kashiwara operators
- Energy $\mathscr{H}$ comes from affine ($\widehat{\mathfrak{sl}}_n$) crystals

Construction of $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ (sketch)

✓ ✓ ✓ ✓



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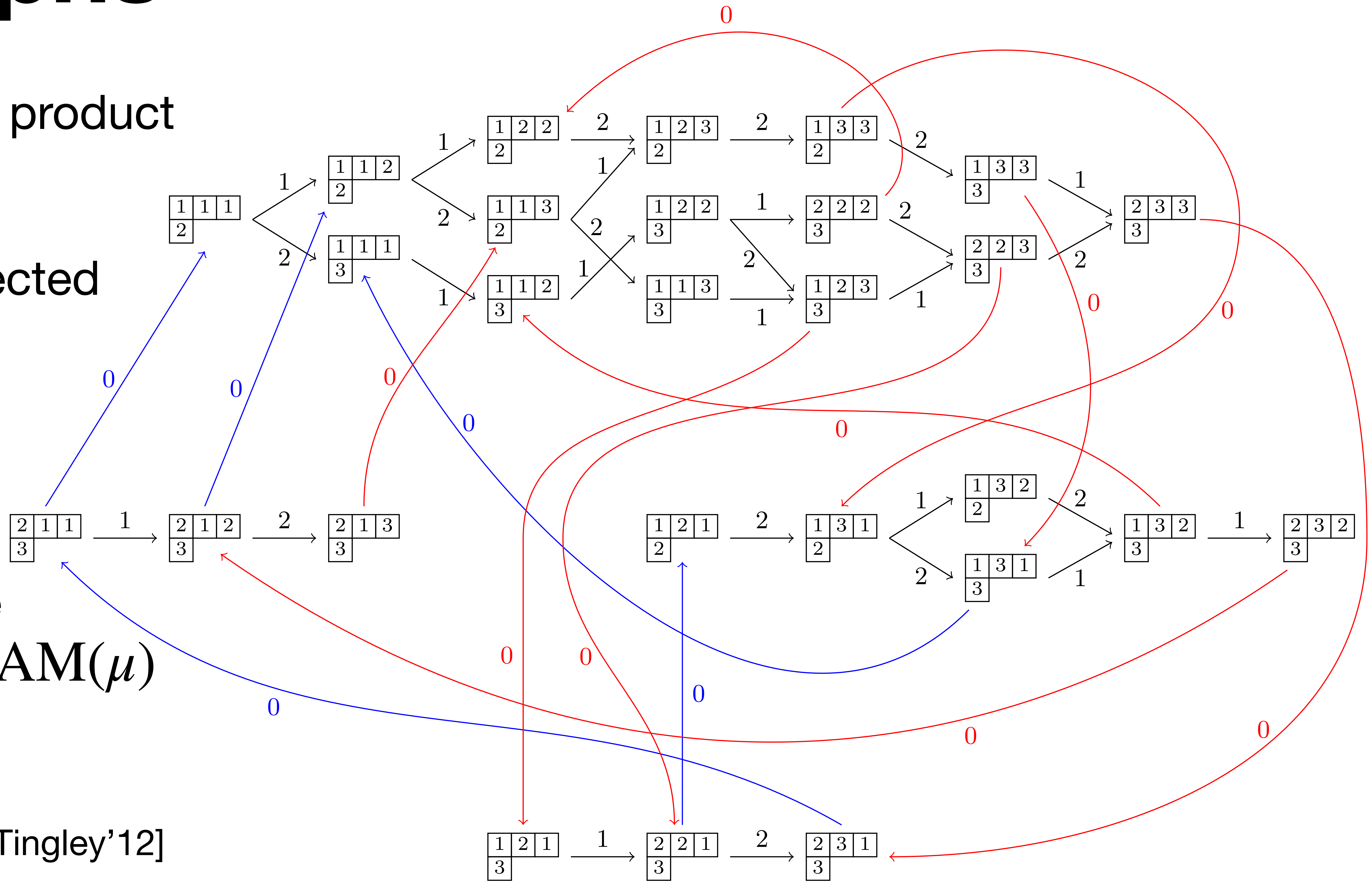
- Classical RSK commutes with $\mathfrak{sl}_n$ Kashiwara operators
- Energy $\mathscr{H}$ comes from affine ($\widehat{\mathfrak{sl}}_n$) crystals
- IDEA : equip (P, Q) and (V, W) of $\widehat{\mathfrak{sl}}_n$ - bicrystal structure preserved by RSK

Affine crystal graphs

- Vertically strict tableaux are tensor product of KR crystals
- The crystal graph $VST(\mu)$ is connected
  [Kashiwara'90]
- Demazure subgraph: remove "bad" $\widetilde{e}_0, \tilde{f}_0$ arrows
- For any V there exist a path on the Demazure subgraph $\mathscr{L}_V : V \mapsto \mathrm{YAM}(\mu)$
- $\mathscr{H}(V) = \#\tilde{f}_0 - \#\widetilde{e}_0$ in $\mathscr{L}_V$
  [Schilling-Tingley'12]



Affine Bicrystal (P, Q)

- On (V, W) bicrystal structure is product structure
- Impose $(P, Q) \mapsto (V, W)$ preserve bicrystal structure
- On (V, W) bicrystal structure is NOT product structure (nontrivial $\widetilde{e}_0, \tilde{f}_0$)
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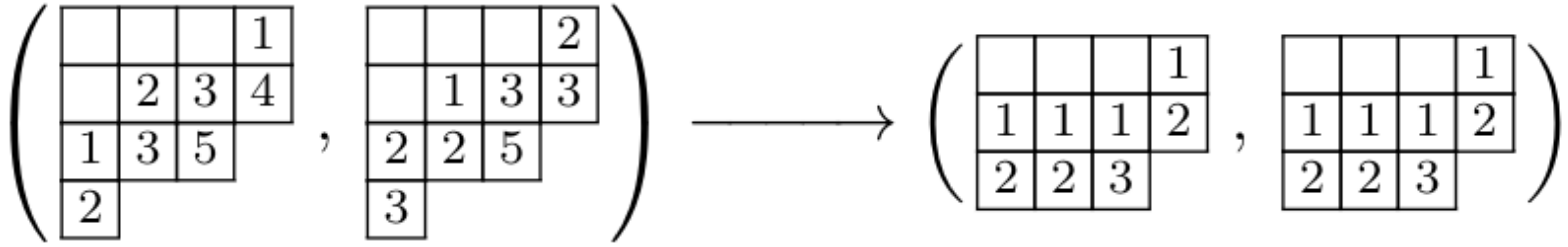

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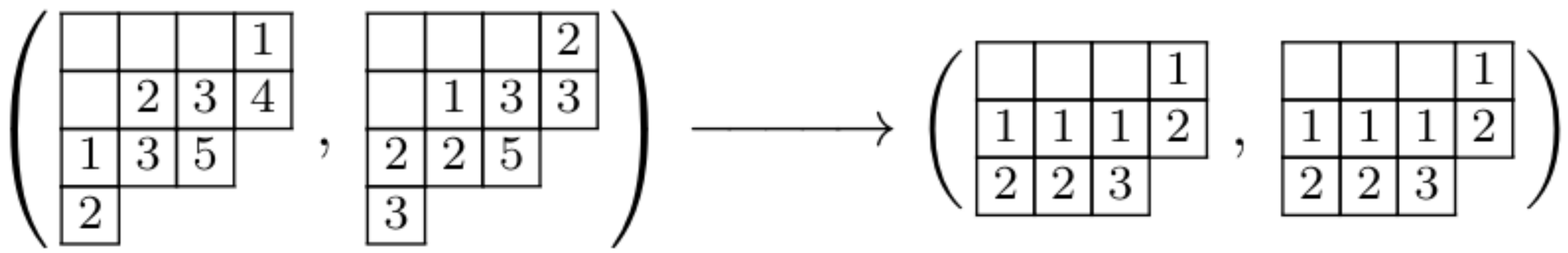

LEMMA (IMS'21)

$\mathscr{L}_{P,Q}$ linearizes the skew RSK map

To sum up

- $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ has the following properties
  - P and V have equal content. Same for Q and W
  - $|\rho| = \mathscr{H}(V) + \mathscr{H}(W) + |\kappa| + |\nu|$ $\quad (\mathscr{H} = \mathscr{H}_1 + \mathscr{H}_2 + \cdots)$
  - $\lambda_1 = \mu_1 + \nu_1$

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Conclusion

- Our bijective theory gives direct connection between solvable non-free fermionic models and positive temperature free fermionic models
- Mysterious appearence of a discrete integrable system (generalization of BBS)
- We construct a bijective q-extension of the RSK correspondence Υ
- With Υ we can prove bijectively the Cauchy identities (CI) for q-Whittaker polynomials (first time)
- With Υ we can prove bijectively refinements of the CI relating q-Whittaker and skew-Schur polynomials