Dynamics on interlacing partitions for sl_2 stochastic vertex models

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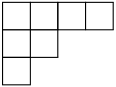
Tokyo Tech

We study statistics on ensembles of Young diagrams: **why** and **how** ?

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0) =$$  (Young diagram)

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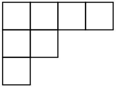


(Young diagram)

- Motivating example: SCHUR PROCESSES

$$s_\lambda(x) = \frac{\det_{i,j=1}^{n}\left(x_i^{\lambda_j+n-j}\right)}{\prod_{i<j}(x_i - x_j)}$$

(Schur functions)

Schur functions are positive polynomials

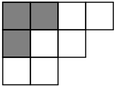
$$s_\lambda(x) = \sum_{T\sim\lambda} x^T$$

$T =$

1	2	2	5
2	3	5	
4			

(semi-standard Young Tableaux)

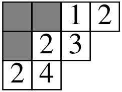
$$x^T = x_1^{\#1} x_2^{\#2} \cdots x_n^{\#n}$$

$\lambda/\mu =$ 

(skew Young diagram)

$$s_{\lambda/\mu}(x) = \sum_{T\sim\lambda/\mu} x^T$$

(skew Schur functions)

$T =$ 

(semi-standard skew Young Tableau)

- Branching rules:

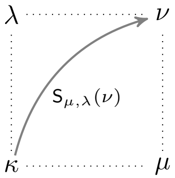
$$s_{\lambda/\mu}(x_1,\ldots,x_n) = \sum_{\nu} s_{\nu/\mu}(x_1,\ldots,x_k)s_{\lambda/\nu}(x_{k+1},\ldots,x_n)$$

- skew Schur functions satisfy:

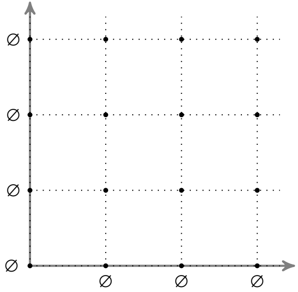
$$\sum_{\nu} s_{\nu/\mu}(x)\, s_{\nu/\lambda}(y) = \Pi(x;y) \sum_{\kappa} s_{\mu/\kappa}(x)\, s_{\lambda/\kappa}(y) \qquad \text{(Cauchy Identity)}$$

- Given partitions μ, λ:

$$\mathsf{S}_{\mu,\lambda}(\nu) = \frac{1}{Z_{\mu,\lambda}} s_{\nu/\mu}(x)\, s_{\nu/\lambda}(y)$$

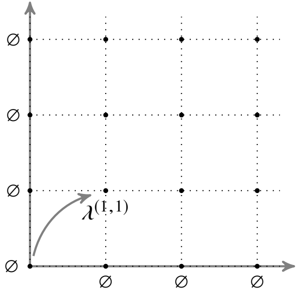


$\mathsf{S}_{\mu,\lambda}(\nu)$ is interpreted as a transition probability from a state κ to a state ν.



(Schur random field)

- We build a random field of partitions $\boldsymbol{\lambda} = \{\lambda^{(i,j)}\}$ using local moves



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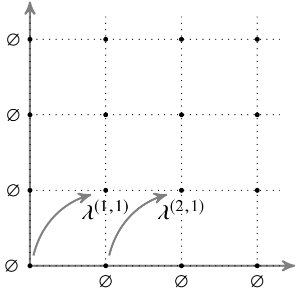

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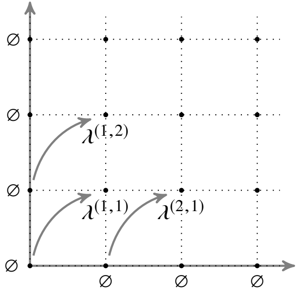

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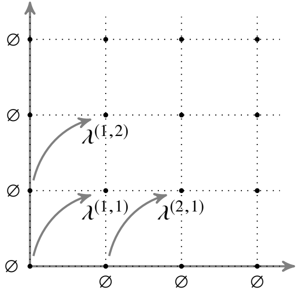

(Schur random field)

- We build a random field of partitions $\boldsymbol{\lambda} = \{\lambda^{(i,j)}\}$ using local moves
- Because of branching rules and skew Cauchy Identities the joint measure along down-right paths is written in terms of Schur functions

$$\mathrm{Prob}(\lambda^{(n,t)} = \lambda) = \frac{s_\lambda(x_1,\ldots,x_n)s_\lambda(y_1,\ldots,y_t)}{\Pi(x;y)}$$

(Schur measure [Okounkov'01])

$$\Pi(x;y) = \prod_{i=1}^{n}\prod_{j=1}^{t}\frac{1}{1-x_i y_j}$$

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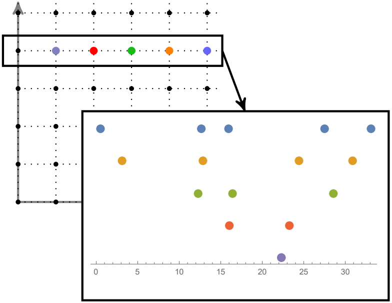
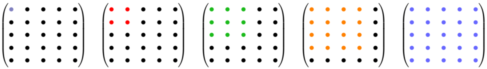


(Schur random field)

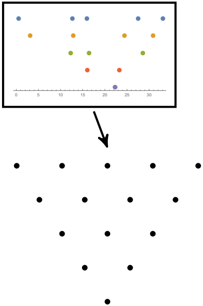
- Partitions on the same row (or column) interlace

$$\lambda^{(1,t)} \prec \lambda^{(2,t)} \prec \cdots \prec \lambda^{(n,t)}$$

- Probability measures on interlacing arrays appear in random matrix theory (eigenvalues of minors of self adjoint matrices)

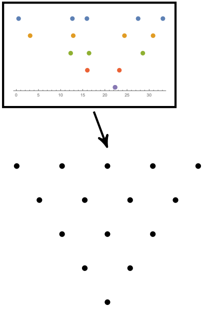


- Schur process [Okounkov-Reshetikhin'03] :



$$\lambda^1 \prec \lambda^2 \prec \cdots \prec \lambda^n$$

$$\frac{s_{\lambda^1}(x_1)\cdots s_{\lambda^n/\lambda^{n-1}}(x_n)\; s_{\lambda^n}(y_1,\ldots,y_t)}{\Pi(x;y)}$$



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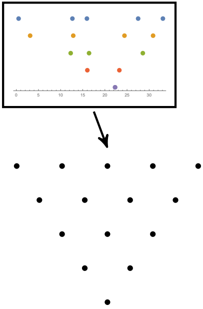
- QUESTIONS
  1. What is the meaning of such process? What do they describe?
  2. Does there exist a more natural way to sample λ?



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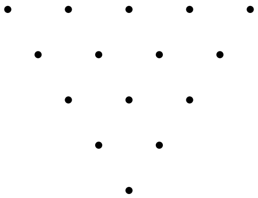
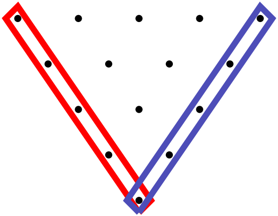
- RELATED PROCESSES: TASEP, PNG, longest increasing subsequence, push-TASEP, eigenvalues of random matrices, etc.
- SAMPLING TECHNIQUES: RSK, Borodin-Ferrari dynamics, Bijectivization of YBE, etc.



(Schur Process)

- A *surprising* fact is that the leftmost and rightmost diagonal evolve as autonomous Markov processes.



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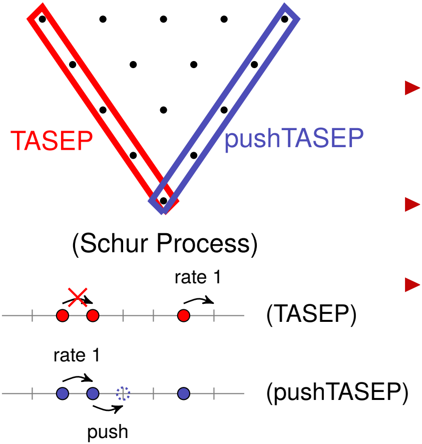

- A *surprising* fact is that the leftmost and rightmost diagonal evolve as autonomous Markov processes.
- Coordinates on the leftmost diagonal sample the TASEP
- Coordinates on the rightmost diagonal sample a pushTASEP

TRAIN OF THOUGHTS

Cauchy Identities for special functions:

$$\sum_\nu s_{\nu/\mu}(x)\, s_{\nu/\lambda}(y) = \Pi(x;y) \sum_\kappa s_{\mu/\kappa}(x)\, s_{\lambda/\kappa}(y)$$

↓

Random sampling of partitions

↓

Marginal processes of the field of random partitions might be interesting (TASEP, pushTASEP,etc.)

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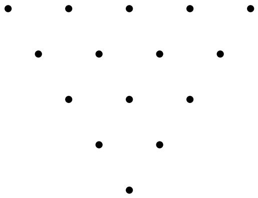
Marginal processes of the field of random partitions might be interesting (TASEP, pushTASEP,etc.)

What processes arise when we consider Cauchy Identities for different special functions?

Replacing the Schur $s_{\lambda/\mu}$ functions with the Macdonald functions $P_{\lambda/\mu}, Q_{\lambda/\mu}$:

$$\sum_{\nu} P_{\nu/\mu}(x)\, Q_{\nu/\lambda}(y) = \Pi(x;y) \sum_{\kappa} P_{\mu/\kappa}(x)\, Q_{\lambda/\kappa}(y)$$

we obtain the *MacDonald Processes* [Borodin-Corwin'11]



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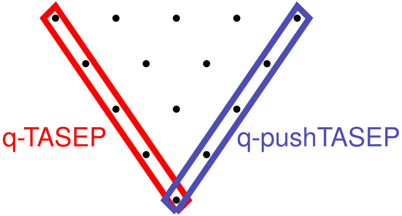

(MacDonald Process)

Diagonals of the process are still markovian.

- q-TASEP and q-pushTASEP can be thought are discretizations of the KPZ equation (stochastic PDE for growth of surfaces with lateral growth and relaxation)
- Algebraic properties (symmetries, operators, etc.) of Macdonald functions allow an exact study of the marginal processes and of the KPZ equation.
- Classical limit:

Macdonald Processes	$\longrightarrow$	Whittaker Processes	$\longrightarrow$	SHE / KPZ
q-TASEP	$\longrightarrow$	Gamma Polymer	$\longrightarrow$	$\partial_t Z = \partial_x^2 Z - \xi Z$

QUESTION:

- What are the most general models that can be studied following the MacDonald Processes scheme?

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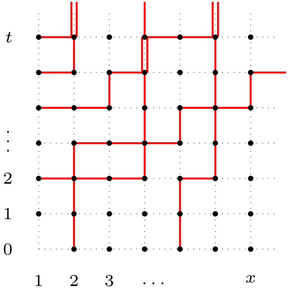
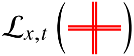
(PARTIAL) ANSWER:

- sl_2 stochastic vertex models (Six Vertex Model [Gwa-Spohn'92],[Borodin-Corwin-Gorin'14], Higher Spin Six Vertex Model [C-Petrov'15], q-Hahn TASEP [Povolotsky'13], q-Hahn PushTASEP [C-Matveev-Pe'18], etc.)

sl_2 Stochastic Vertex Models:



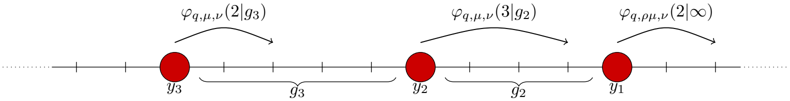
- Probability of configuration of red path = product vertex weights



Vertex weights depend on many parameters (q, s, u_t, θ_x) and they satisfy the Yang-Baxter equation.

Example: q-Hahn TASEP

$$\varphi_{q,\mu,\nu}(k|n) = \mu^k \frac{(\nu/\mu;q)_k(\mu;q)_{n-k}}{(\nu;q)_n} \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$$



- $\varphi_{q,\mu,\nu}$: q-deformed Beta-binomial
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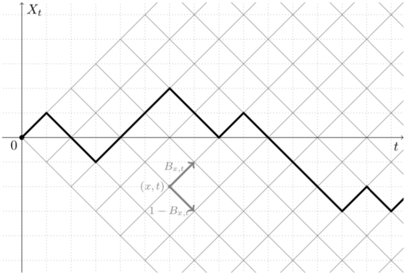


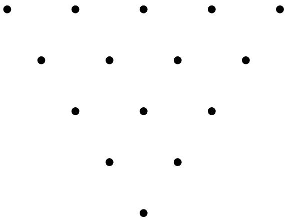
- $\varphi_{q,\mu,\nu}$: q-deformed Beta-binomial
- generalization of q-TASEP
- generalization of directed random walks in Beta random environment (fig. from [Barraquand-Corwin'15])

RESULTS [Bufetov-M-Petrov'19,M-Petrov'??]

- We build a random field of partitions using $\mathbb{F}_{\lambda/\mu}$: spin q-Whittaker functions [Borodin-Wheeler'17]

$$\mathbb{F}_{\lambda/\mu}(x) = x^{|\lambda|-|\mu|} \prod_{i=1}^{\ell(\lambda)-1} \frac{(-s/x;q)_{\lambda_i-\mu_i}(-sx;q)_{\mu_i-\lambda_{i+1}}(q;q)_{\lambda_i-\lambda_{i+1}}}{(q;q)_{\lambda_i-\mu_i}(q;q)_{\mu_i-\lambda_{i+i}}(s^2;q)_{\lambda_i-\lambda_{i+i}}}$$

$$\sum_\nu \mathbb{F}_{\nu/\lambda}(x)\,\mathbb{F}^*_{\nu/\mu}(y) = \frac{(-sx;q)_\infty(-sy;q)_\infty}{(s^2;q)_\infty(xy;q)_\infty} \sum_\varkappa \mathbb{F}_{\mu/\varkappa}(x)\mathbb{F}^*_{\lambda/\varkappa}(y).$$



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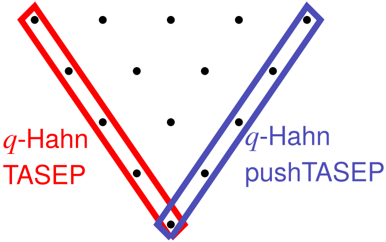

($\mathbb{F}/\mathbb{F}^*$ Process)

Technical points that we address:

- proof that diagonals are autonomous Markov processes
- initiate theory of operators for functions F, $\mathbb{F}$
- give exact expressions to average of observables of the process

RESULTS [Bufetov-M-Petrov'19,M-Petrov'??]

- Operators:

$$\mathfrak{D}_1^N = \sum_{i=1}^N \prod_{\substack{j=1 \\ j\neq i}}^N \frac{(1+sx_i)}{1-x_i/x_j} T_{q,x_i}, \qquad \overline{\mathfrak{D}}_1^N = \sum_{i=1}^N \prod_{\substack{j=1 \\ j\neq i}}^N \frac{(1+s/x_i)}{1-x_j/x_i} T_{q^{-1},x_i}.$$

$$\mathfrak{D}_1^N \mathbb{F}_\lambda(x_1,\dots,x_N) = q^{\lambda_N} \mathbb{F}_\lambda(x_1,\dots,x_N).$$

$$\overline{\mathfrak{D}}_1^N \mathbb{F}_\lambda(x_1,\dots,x_N) = q^{-\lambda_1} \mathbb{F}_\lambda(x_1,\dots,x_N).$$

- Observables:
  - λ_N: current in particle system / position random walkers / height KPZ
  - λ_1: current in particle system / partition function Beta polymer / height KPZ

Summary of the talk

1. Probability on interlacing partitions: Schur processes and random symmetric matrices
2. Non free fermionic models: MacDonald processes
3. Taking the scheme to a more general level: sl_2 stochastic vertex models and spin q-Whittaker processes

OPEN QUESTIONS

- complete the theory of operators of $\mathbb{F}$ functions (we only got two)
- study of the full $\mathbb{F}/\mathbb{F}^*$ process (not only diagonals)
- clearer connection between $\mathbb{F}/\mathbb{F}^*$ process and Random Walkers in Random environment